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EXISTENCE OF HOMOTOPIC HARMONIC MAPS INTO METRIC SPACE OF NONPOSITIVE CURVATURE

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ABSTRACT. The definitions and techniques, which deals with homotopic harmonic maps from a compact Riemannian manifold into a compact metric space, developed by N. J. Korevaar and R. M. Schoen [7] can be applied to more general situations. In this paper, we prove that for a complicated domain, possibly noncompact Riemannian manifold with infinitely generated fundamental group, the existence of homotopic harmonic maps can be proved if the initial map is simple in some sense.

1. Introduction

The theory of harmonic maps is a kind of optimization problem. So the existence of such maps is of basic importance in the theory of harmonic maps. In this paper we study the existence of harmonic(locally energy minimizing) map which is homotopic to a given map into a metric space.

There are several results on the existence of harmonic maps between Riemannian manifolds following the work of J. Eells and J. H. Sampson [3]. Recently, people have become interested in the analysis on metric spaces(or on the Alexandrov space that are Gromov-Hausdorff limits of Riemannian manifolds) and there have been some progress in developing generalized notions of energy and harmonicity (see [5], [6], [7]).

Korevaar-Schoen[7] generalized the Sobolev theory for maps from a Riemannian manifold into a metric space and defined the energy by using the convergence of the ε energy density measure. In [7], the following three kinds of existence theorems have been proved for harmonic maps into metric spaces. They defined nonpositively curved metric

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space(NPC) using the triangle comparison in global sense. We, however, see the curvature as a local property and will use the notion of CAT(0) space as defined in [2].

THEOREM A (DIRICHLET PROBLEM; [7]). *Let* $(\Omega,\ g)$ *be a Lipschitz Riemannian domain (a relatively compact, connected, open subset with Lipschitz boundary of a Riemannian manifold) and let* (X, d) *be a CAT(0) metric space. Then for any* $\phi \in W^{1,2}(\Omega, X)$, *there exists a unique* $u \in W^{1,2}(\Omega, X)$ *which is stationary for the p=2 Sobolev energy with the property* $tr(u) = tr(\phi)$.

THEOREM B (EQUIVARIANT MAPPING PROBLEM; [7]). *Let* M *be a complete Riemannian manifold with finite volume and* $\tilde{M}$ *be the universal covering mainfold of* M *and let* X *be a CAT(0) metric space. Let* $\Gamma = \pi_1(M)$ *be finitely generated and* $\rho : \Gamma \to isom(X)$ *be a homomorphism. Then there exists an equivariant minimizing sequence* $\{u_i : \tilde{M} \to X\}$ *which has local modulus of continuity control. That is, for each* $x \in \tilde{M}$ *we assume there is an (equivariant) function* $\omega(x, r)$ $(0 \le r \le r_x)$ *which is monotone increasing in* r, *which satisfies* $\omega(x, 0) = 0$ *and so that*

$$\sup_i \sup_{|x-z|\le r} d(u_i(x), u_i(z)) \le \omega(x, r)$$

Furthermore the sequence $\{u_i\}$ *converges (locally uniformly and hence in* L^2_{loc}*) to an equivariant harmonic map* u *if and only if there exists an* $x \in \tilde{M}$ *at which the sequence of points* $\{u_i(x)\}$ *is convergent.*

Using Theorem B, they constructed an equivariant minimizing sequence $\{u_i : M \to N\}$, which is equicontinuous, from a compact Riemannian manifold M to a compact metric space of nonpositive curvature N and proved the following theorem by the existence of Dirichlet problem(Theorem A).

THEOREM C (HOMOTOPY PROBLEM; [7]). *Let* M *be a compact Riemannian manifold without boundary and let* N *be a compact inner metric space with CAT(0) universal cover* X. *Then for any continuous map* $f : M \to N$ *there exists a Lipschitz harmonic map* $u : M \to N$ *which is homotopic to* f.

It turns out that the similar techniques can be applied to more general situations. In fact, we can consider the equivariant maps defined on a regular cover $\tilde{M}$ of M with the action of quotient group $\pi_1(M)/\pi_1(\tilde{M})$ which is finitely generated, and construct an equivariant minimizing sequence $\{u_i : \tilde{M} \to X\}$ which has local modulus of continuity control and the same criterion of convergence.

Using this equivariant minimizing sequence $\{u_i : \tilde{M} \to X\}$, we can prove the following more general result on the existence of harmonic maps from a Riemannian manifold(possibly noncompact) to a compact metric space of nonpositive curvature.

MAIN THEOREM. *Let M be a complete Riemannian manifold with finite volume, without boundary and with a normal subgroup $H \lhd \pi_1(M)$ so that $\Gamma = \pi_1(M)/H$ is finitely generated. Let N be a compact geodesic space of nonpositive curvature.*

Then for each continuous map $f : M \to N$ with $f_\sharp(H) = 0$, there exists a Lipschitz harmonic map $u : M \to N$ which is homotopic to f.

2. Preliminaries

The energy E of a map $u : M \to N$ between two Riemannian manifolds M and N is defined by

$$E^u = \int_M |du|^2 d\mu,$$

where $|du|$ is the Hilbert-Schmidt norm of the differential $du \in T^*M \otimes u^*TN$ and $d\mu$ is the measure given by the metric on M.

When we think of a map with metric space target, the above definition is no more valid. The difficulty is in the fact that the target space N has no differential structure.

There are some generalizations of the energy (see [5], [6], [7]). Among them, we recall the definitions in [7].

Let M be a complete Riemannian manifold and (N, d) be a complete metric space. The space $L^2(M, N)$ is defined by

$$L^2(M, N) = \{u : M \to N | \int_M d^2(u(x), Q) d\mu(x) < \infty \ for\ some\ Q \in N\}.$$

Then $L^2(M,N)$ is a complete metric space with the distance function D defined for each $u, v \in L^2(M,N)$ by

$$D^2(u,v) = \int_M d^2(u(x), v(x))d\mu(x).$$

For a map $u \in L^2(M,N)$ and $\varepsilon > 0$, define

$$e_\varepsilon(x,y) = \frac{d^2(u(x), u(y))}{\varepsilon^2},$$

where $(x,y) \in M \times M$. Averaging $e_\varepsilon(x,y)$ spherically and radially, we can define $\varepsilon -$ *approximate energy density*

$${}_\nu e_\varepsilon(x) = \int_0^2 \int_{S(x,\lambda\varepsilon)} e_{\lambda\varepsilon}(x,y) \frac{d\sigma_{x,\lambda\varepsilon}(y)}{(\lambda\varepsilon)^{n-1}} d\nu(\lambda),$$

where $S(x,\lambda\varepsilon)$ is the sphere centered at x of radius $\lambda\varepsilon$, $d\sigma_{x,\lambda\varepsilon}$ is the surface measure on $S(x,\lambda\varepsilon)$ and ν is a Borel measure on the interval $(0,2)$ satisfying

$$\nu \geq 0\,,\ \nu((0,2)) = 1\,,\ \int_0^2 \lambda^{-2} d\nu(\lambda) < \infty.$$

The $\varepsilon -$ *energy functional* ${}_\nu E_\varepsilon : C_c(M) \to \mathbb{R}$ is defined by

$${}_\nu E_\varepsilon(f) = \int_M f(x)\,{}_\nu e_\varepsilon(x) d\mu(x)\,,\quad f \in C_c(M)$$

The *Sobolev space* $W^{1,2}(M,N)$ is defined by

$$W^{1,2}(M,N) = \{u \in L^2(M,N) | {}_\nu E^u < \infty\},$$

where

$${}_\nu E^u = \sup_{\substack{f \in C_c(M) \\ 0 \leq f \leq 1}} \left(\limsup_{\varepsilon - 0} E_\varepsilon^u(f) \right).$$

The *Sobolev space* $W^{1,2}(M,N)$ is a metric space as a subspace of $L^2(M, N)$. i.e. the topology of $W^{1,2}(M,N)$ is given by the distance D. It was shown that for $u \in W^{1,2}(M,N)$ each measure ${}_\nu e_\varepsilon(x)d\mu(x)$ converges weakly to the same *energy density measure de* having total mass ${}_\nu E^u$ so that

$$E^u(f) = {}_\nu E^u(f) = \lim_{\varepsilon \to 0} {}_\nu E_\varepsilon(f) \text{ for all } f \in C_c(M).$$

Furthermore, the energy density measure de is absolutely continuous with respect to Lebesgue measure $d\mu$ so that

$$de(x) = |\nabla u|_2(x)d\mu(x)$$

for some function $|\nabla u|_2 \in L^1(M, R)$. The *energy functional* $E^u : C_c(M) \to \mathbb{R}$ is defined by the measure

$$|\nabla u|^2(x)d\mu(x) = \frac{1}{\omega_n}|\nabla u|_2(x)d\mu(x),$$

where $\omega_n = vol(S^{n-1})$ for the consistence with the Riemannian case. The *energy* E^u of u is defined by

$$E^u = \int_M |\nabla u|^2(x)d\mu(x)$$

if it is finite.

Now we recall the definitions of CAT(0) and nonpositively curved metric space (ref. [2], [4], [8]). A metric space (X,d) is said to be a *geodesic metric space* if every pair of points in X can be joined by a distance realizing curve, i.e. a curve with length equal to the distance between two end points. A *geodesic segment* in X is a unit speed parametrization of such a curve. A *geodesic triangle* Δ in X is a triple of points P, Q, $R \in X$ together with a choice of three geodesic segments, one joining each pair of vertices. A *comparison triangle* for Δ is a triangle $\bar{\Delta}$ in the Euclidean plane E^2 with vertices $\bar{P}$, $\bar{Q}$, $\bar{R}$ such that $d(P,Q) = d(\bar{P},\bar{Q})$, $d(Q,R) = d(\bar{Q},\bar{R})$, $d(R,P) = d(\bar{R},\bar{P})$(such an Euclidean triangle is unique up to isometry). A length space X is said to satisfy the CAT(0) inequality globally (or 'X is said to be a CAT(0) space') if and only if for any three points P, Q, R in X and choices of geodesics $\gamma_{P,Q}$ (of length r), $\gamma_{Q,R}$ (of length p), $\gamma_{R,P}$ (of length q), connecting the respective points, the following comparison property is to hold:

For any $0 < \lambda < 1$, write Q_λ for the point on $\gamma_{Q,R}$ which is a fraction λ of the distance from Q to R. That is,

$$d(Q_\lambda, Q) = \lambda p, \ d(Q_\lambda, R) = (1-\lambda)p.$$

On the comparison triangle of side lengths p, q, r and opposite vertices $\bar{P}$, $\bar{Q}$, $\bar{R}$, there is a corresponding point

$$\bar{Q}_\lambda = \bar{Q} + \lambda(\bar{R} - \bar{Q}).$$

such that the metric distance $d(P, Q_\lambda)$ is bounded above by the Euclidean distance $|\bar{P} - \bar{Q}_\lambda|$. i.e.

$$d(P, Q_\lambda) \leq |\bar{P} - \bar{Q}_\lambda|$$

The nonpositive curvature condition is a local property. That is, a geodesic metric space X is of nonpositive curvature if and only if the CAT(0) inequality holds locally, i.e. for each point $P \in X$ there is a neighborhood U of P such that the CAT(0) inequality hold for any triangle in U. For example, the circle and net is of nonpositive curvature but not CAT(0) because they are not contractible.

CAT(0) space is a space of nonpositive curvature and complete simply connected geodesic metric space of nonpositive curvature is a CAT(0) space(see [1]). The covering space of a geodesic metric space of nonpositive curvature is also a geodesic metric space of nonpositive curvature. So the universal covering space of a geodesic metric space of nonpositive curvature is CAT(0).

3. Existence results

Let M be a complete Riemannian manifold, possibly with smooth compact boundary ∂M and let $\tilde{M}$ be a regular covering manifold of M

with the projection $p : \tilde{M} \to M$. In other words $p_\sharp\pi_1(\tilde{M})$ is a normal subgroup of the fundamental group $\pi_1(M)$ where $p_\sharp : \pi_1(\tilde{M}) \to \pi_1(M)$ is the induced homomorphism from p. Then the quotient group $\Gamma = \pi_1(M)/p_\sharp\pi_1(\tilde{M})$ acts on $\tilde{M}$ properly discontinuously and freely.

Let X be a complete metric space and $\rho : \Gamma \to isom(X)$ a homomorphism. A map $u : \tilde{M} \to X$ is said to be $\rho-$ *equivariant* if

$$u(\gamma x) = \rho(\gamma)u(x) \text{ for all } x \in \tilde{M} \text{ and } \gamma \in \Gamma.$$

An equivariant map u is said to be *harmonic* if it is (locally) a Sobolev map and if it is stationary for the energy(locally energy minimizing) defined for locally Sobolev, equivariant $v : \tilde{M} \to X$ by

$$E^v = \int_M |\nabla v|^2 d\mu.$$

This integral is well defined from the Γ-invariance of the energy density as long as M has finite volume.

For the existence of equivariant harmonic maps in this situation, one can use the direct method. In case $\tilde{M}$ is the universal covering manifold and Γ is finitely generated, Korevaar-Schoen showed the existence of ρ-equivariant, locally Lipschitz map $u : \tilde{M} \to X$ using the center of mass construction. More generally, when $\tilde{M}$ is a regular covering manifold we can prove the the existence of ρ-equivariant, locally Lipschitz map using the center of mass construction.

PROPOSITION 3.1. *Let M, $\tilde{M}$, Γ, ρ be as above with Γ being finitely generated, $\partial M = \phi$ and let X be a CAT(0)-space(ref. [1]). Then there exists a ρ-equivariant, locally Lipschitz map $u : \tilde{M} \to X$ with the locally bounded Lipschitz constant.*

Moreover, by the analoguous argument as in [7], we can prove the existence of energy minimizing, Lipschitz, ρ-equivariant sequence $\{u_i\}$ defined on a regular covering manifold $\tilde{M}$.

THEOREM 3.2. *Let M be a complete Riemannian manifold with finite volume without boundary and $p : \tilde{M} \to M$ be a regular covering of M such that $\Gamma = \pi_1(M)/p_\sharp\pi_1(\tilde{M})$ is finitely generated. Let X be a*

CAT(0)-space and $\rho : \Gamma \to isom(X)$ be a homomorphism. Suppose the set of finite energy ρ-equivariant maps from $\tilde{M}$ to X is nonempty.

Then there is an equivariant minimizing sequence $\{u_i : \tilde{M} \to X\}$, so that for any compact subset $K \subset M$ and i sufficiently large (depending on K), the u_i is Lipschitz continuous on (the lift in $\tilde{M}$ of) K.

REMARK 3.3. In case of equivariant maps on regular covering manifold, the same criterion holds for the convergence of Lipschitz equivariant minimizing sequence as in [7]. In fact, the convergence at one point guarantee the convergence of the sequence of maps because it is an equicontinuous family.

Now we consider the homotopy problem. Let M be a complete Riemannian manifold with finite volume and N be a geodesic metric space. A continuous map $u : M \to N$ is said to be *harmonic* if it is locally energy minimizing. Precisely, 'locally energy minimizing' means that for each $x \in M$, there is a neighborhood of x such that all $W^{1,2}$-comparison maps which agree with u outside this neighborhood have no less energy.

Eells-Sampson[3] proved in 1964 that in case M is a compact Riemannian manifold and N is a Riemannian manifold of nonpositive sectional curvature, there is a harmonic map in each homotopy class of maps from M to N. And Korevaar-Schoen generalized that to the case of compact Riemannian manifold domain and NPC(space of nonpositive curvature) target.

We prove the existence of homotopic harmonic maps in more general case of noncompact domain.

MAIN THEOREM. *Let M be a complete Riemannian manifold with finite volume, without boundary and with a normal subgroup $H \lhd \pi_1(M)$ so that $\Gamma = \pi_1(M)/H$ is finitely generated. Let N be a compact geodesic space of nonpositive curvature.*

Then for each continuous map $f : M \to N$ with $f_\sharp(H) = 0$, there exist a Lipschitz harmonic map $u : M \to N$ which is homotopic to f.

REMARK 3.5. If the domain M is compact then $\pi_1(M)$ is finitely generated, so the normal subgroup H can be chosen to be 0. Hence for any continuous map $f : M \to N$, there exist a Lipschitz harmonic map $u : M \to N$ homotopic to f. This is the case discussed in [7].

Before proving the theorem, we state some lemmas.

Lemma 3.6. *Let Γ be a Lie group acting on a Riemannian manifold M, X be a CAT(0)-space and let $\rho : \Gamma \to isom(X)$ be a homomorphism. Then any two continuous $W^{1,2}$ map u, $v : M \to X$ are homotopic in $W^{1,2}$ via the ρ-equivariant homotopy given by geodesics in X.*

Proof. Given two continuous u, $v \in W^{1,2}(M,X)$ define a homotopy $F : M \times [0,1] \to X$ by the following.

Since X is a CAT(0)-space, there is a unique geodesic $\alpha_x : [0,1] \to X$ from $u(x)$ to $v(x)$ for each $x \in M$. Now define $F(x,t) = \alpha_x(t)$ for each $(x,t) \in M \times [0,1]$, then we can easily see that F is continuous in both variable from the convexity of CAT(0)-space. Furthermore, for each $t \in [0,1]$, $u_t = F(\cdot,t)$ belongs to $W^{1,2}(M,X)$ by the energy convexity;

$$E^{u_t} \leq tE^u + (1-t)E^v - t(1-t)\int_M |\nabla d(u,v)|^2.$$

So F is well defined.

For the equivariance of F, we need to show the following;

$$u_t(\gamma x) = \rho(\gamma)u_t(x) \quad \text{for each } \gamma \in \Gamma, (x,t) \in M \times [0,1].$$

Since the isometry $\rho(\gamma)$ moves geodesic to geodesic, $\beta(t) = \rho(\gamma)u_t(x)$ is the geodesic from $\rho(\gamma)u(x) = u(\gamma x)$ to $\rho(\gamma)u_1(x) = v(\gamma x)$. On the other hand, the curve $\delta(t) = u_t(\gamma x)$ is also a geodesic from $\rho(\gamma)u(x) = u(\gamma x)$ to $\rho(\gamma)u_1(x) = v(\gamma x)$. By the uniqueness of geodesics in CAT(0)-space, we can conclude that β and δ coinside. Hence $u_t(\gamma x) = \rho(\gamma)u_t(x)$ and the proof is complete. □

Lemma 3.7. ([7] p.54). *Let (Ω, g) be a Riemannian domain, a connected relatively compact open subset of a Riemannian manifold (M,g) with Lipschitz boundary, and let (X,d) be a CAT(0) metric space. For a map $\phi \in W^{1,2}(\Omega, X)$ define*

$$W^{1,2}_\phi = \{u \in W^{1,2}(\Omega, X) | tr(u) = tr(\phi)\},$$

where $tr(u)$ denotes the trace of u on the boundary $\partial\Omega$ of Ω.

Then there exists a unique $u \in W^{1,2}_\phi$ which is minimizing for the Sobolev energy. In fact the energy $E^u = \int_\Omega |\nabla u|^2 d\mu$ of u satisfies

$$E^u = E_0 := \inf_{v \in W^{1,2}_\phi} E^v.$$

Now we are ready to prove the Main Theorem.

PROOF OF THE MAIN THEOREM. By the correspondence between conjugacy classes of subgroups of $\pi_1(M)$ and the covering spaces over M, there is a covering space $\bar{M}$ with the covering map $p : \bar{M} \to M$ such that $p_\sharp(\pi_1(\bar{M})) = H$. Since H is a normal subgroup of $\pi_1(M)$, $\bar{M}$ is a regular covering space. Furthermore, $\bar{M}$ is a Riemannian manifold.

It is easy to see that $\Gamma = \pi_1(M)/H$ acts on $\bar{M}$ freely and $\pi_1(N)$ acts on the universal covering space X of N isometrically.

For a fixed $x \in M$ and $\bar{x} \in p^{-1}(x)$, f induces the homomorphism

$$\tilde{f}_\sharp : \pi_1(M)_x / p_\sharp(\pi_1(\bar{M})_{\bar{x}}) \to \pi_1(N)_{f(x)}$$

because $f_\sharp(p_\sharp(\pi_1(\bar{M})_{\bar{x}})) = 0$. Now we can choose a lifting $\bar{f} : \bar{M} \to X$ so that $\bar{f}$ is $\tilde{f}_\sharp$-equivariant.

Since $f \in W^{1,2}(M,N)$, $\bar{f} \in W^{1,2}(\bar{M},X)$. In other words, there is a $\tilde{f}_\sharp$-equivariant $W^{1,2}$-map. So by the Theorem 2.3 we can construct a $\tilde{f}_\sharp$-equivariant energy minimizing sequence $\{\bar{u}_i : \bar{M} \to X\}$ so that for any compact subset $K \subset M$ and i sufficiently large (depending on K), the $\bar{u}_i$ is Lipschitz continuous on (the lift in $\bar{M}$ of) K.

By Lemma 3.3 $\bar{u}_i$'s are homotopic to $\bar{f}$ via $\tilde{f}_\sharp$-equivariant homotopies. So the projections u_i's are homotopic to f and for each compact subset $K \subset M$ and sufficiently large i, u_i is uniformly Lipschitz continuous on K. Since N is compact and the family $\{u_i\}$ is equicontinuous, a subsequence converges uniformly to a limit map on each compact $K \subset M$.

Now we exhaust M by a nested sequence of compact subsets $\{K_n\}$

$$\text{i.e. } K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \quad \text{and}$$

$$M = \bigcup_{i=1}^{\infty} K_i .$$

For K_1, there is a subsequence $\{u_i^1\}$ of $\{u_i\}$ which converges uniformly to a limit map v_1 on K_1.

For K_2, there is also a subsequence $\{u_i^2\}$ of $\{u_i^1\}$ which converges uniformly to a limit map v_2. Then v_2 coinsides with v_1 in K_1. Continuing this process, one can obtain uniformly convergent subsequences $\{u_i^n\}$ for each n and the uniform limit v_n with the property

$$v_n|_{K_i} = v_i \quad \text{for each } i \leq n .$$

Clearly the sequence $\{v_n\}$ converges uniformly on each compact subsets to a limit map $u : M \to N$.

Now it remains to show that u is harmonic. Note that each v_n is locally energy minimizing in K_n by the analoguous argument as in [7]. Since every compact subset of M is contained in some K_n and u coinside with u_n in K_n, u is locally energy minimizing. The proof is complete. □

References

1. Alexander, S. B., Bishop, R. L., *The Hadamard-Cartan Theorem in Locally convex Metric Space*, Enseign. Math. **36** (1990), 309-320.
2. Bridson, Martin R., *On the existence of flat planes in spaces of nonpositive curvature*, preprint (1994).
3. Eells, J., Sampson, J. H., *Harmonic Mappings of Riemannian Manifolds*, Amer. J. Math., vol. **86** (1964), 109-160.
4. Gromov, M., *Hyperbolic groups*, vol. 8, in "Essays in group theory"(S.M. Gersten, ed.), MSRI Publ., Springer Verlag, New York, 1985
5. Gromov, M., Schoen, R. M., *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, preprint.
6. Jost, J., *Equilibrium Maps between Metric Spaces*, Calc. Var., vol. 2, 1994, pp. 173-204.
7. Korevaar, N. J., Schoen, R. M., *Sobolev Spaces and Harmonic Maps for Metric space Targets*, First MSJ International Research Institute on Geometry and Global Analysis **2** (1993), 1-83.
8. Shiohama, K., *An Introduction to the Geometry of Alexandrov Spaces*, Proc. Workshop in Pure Math.(Daewoo), vol. 12, Seoul Korea, 1992.

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