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ELEMENTARY PROOF OF THE NONEXISTENCE OF NODAL SOLUTIONS FOR SOME QUASILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. Consider the problem $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u$ in B, $u = 0$ on ∂B; where $B \subset \mathbb{R}^n$ is a ball, $\lambda > 0$, $1 < p < n$ and $p^* = \frac{np}{n-p}$ is the critical Sobolev exponent. For given $\lambda > 0$, we show that there exists $k = k(\lambda) \in \mathbb{N}$ such that any radial solutions to this problem have at most k nodal curves when $p \leq q \leq p^* - 1$.

Let B_r be the open ball of radius $r > 0$ with its center at the origin $\mathbb{R}^n$ with $n > 1$. We are concerned with the nonexistence of changing sign solutions of the quasilinear problem

$$(\mathrm{Q}_\lambda) \qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where $1 < p < n$, $p^* = \frac{np}{n-p}$, $p \leq q < p^*$ and λ is a real parameter. When $p = q = 2$ in (Q_λ), Atkinson-Brezis-Peletier [1] proved the nonexistence of solutions which change sign, namely, nodal solutions. Later Jones [3] studied the same problem for $p = 2 \neq q$. Recently, Filippucci-Ricci-Pucci [2] extended these nonexistence results to the quasilinear problem (Q_λ) and obtained Theorem 1 a) below. They considered (Q_λ) as ordinary differential equations and used the techniques in the field. In this paper, we present a more direct and elementary proof of Theorem 1 a), by using the same ideas as in [6] and also deduce that for fixed $\lambda > 0$, the number of nodes of any radial solutions to this problem is bounded.

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THEOREM 1. *Suppose* $p \le q \le p^*-1$ *(i.e.* $n \le \frac{p(q+1)}{q-p+1}$*).* $p \le q \le p^*-1$, *that is,* $p < n \le \frac{p(q+1)}{q-p+1}$. *Then a) there exists a constant* $\lambda^* > 0$ *such that* (Q_λ) *has no radial nodal solutions if* $\lambda \in]0, \lambda^*[$ *;*
b) for each $\lambda > 0$ *there exists* $k = k(\lambda) \in \mathbb{N}$ *such that if* u *is a radial solution of* (Q_λ) *then the number of nodes of* u *is less than* $k(\lambda)$.

PROOF. Suppose $w(x)$ is a radial solution of (Q_λ) and $w(x)$ changes sign. Then for some a with $0 < a < 1$, $w(x)$ can be decomposed into

$$w(x) = \begin{cases} u(x), & \text{if } x \in B_a \\ v(x), & \text{if } x \in B_1 \backslash B_a, \end{cases}$$

where $u(x) = u(|x|)$ may be assumed to be a positive radial solution of (Q_λ) in B_a and $v(x) = v(|x|)$ a radial solution of (Q_λ) in $B_1 \backslash B_a$. Moreover, $u'(a) = v'(a)$. By the Pohožaev identity [4],

$$\frac{p-1}{p}\omega_n a^n |u'(a)|^p = \alpha\lambda \int_{B_a} u^q, \tag{1.1}$$

where w_n is the surface area of the unit sphere in $\mathbb{R}^n$ and $\alpha = \frac{n}{q} - \frac{n-p}{p}$. Integrating (Q_λ) in B_a we obtain

$$\omega_n a^{n-1}|u'(a)|^{p-1} = \int_{B_a} \{u^{p^*-1} + \lambda u^{q-1}\}. \tag{1.2}$$

Combining (1.1) and (1.2) we have

$$\int_{B_a} \{u^{p^*-1} + \lambda u^{q-1}\} = a^{\frac{n-p}{p}} \omega_n^{\frac{1}{p}} \left[\frac{\lambda\alpha p}{p-1} \int_{B_a} u^q\right]^{\frac{p-1}{p}}. \tag{1.3}$$

Using Hölder's inequality and (1.3) we obtain

$$\int_{B_a} u^q \le (\frac{\omega_n}{n} a^n)^{\frac{p^*-q-1}{p^*-1}} \left[\int_{B_a} u^{p^*-1}\right]^{\frac{q}{p^*-1}}$$

and

$$\int_{B_a} \{u^{p^*-1} + \lambda u^{q-1}\} \le C \left[\frac{\lambda\alpha p}{p-1}\right]^{k_1} a^{k_2},$$

where $k_1 = \frac{(p-1)(p^*-1)}{p(p^*-1)-q(p-1)}$, $k_2 = \frac{p(n-1)(p^*-1)-qn(p-1)}{p(p^*-1)-q(p-1)}$ and $C = C(n)$ is some constant. From (1.2), we obtain an estimate

$$|u'(a)| \le C\lambda^{\frac{p^*-1}{p(p^*-1)-q(p-1)}} a^{\frac{-q}{p(p^*-1)-q(p-1)}}. \tag{1.4}$$

Now, we consider $v(x)$. Suppose $v(r)$ attains its infimum at $r = \tau$. For $r \in [a, \tau]$,

$$|v'(r)|^{p-1} = r^{1-n} \int_r^\tau (|v|^{p^*-1} + \lambda |v|^{q-1})\, s^{n-1} ds. \tag{1.5}$$

Considering $r = a$ in (1.5), we have $|v'(r)| \le (\frac{a}{r})^{\frac{n-1}{p-1}} |v'(a)|$ and

$$|v(\tau)| \le \int_a^\tau |v'(r)| dr \le \frac{p-1}{n-p} a |v'(a)|. \tag{1.6}$$

Taking $C_1,\ C_2 > 0$ such that $|v|^q \le C_1 |v|^p + C_2 |v|^{p^*}$, we have

$$\begin{aligned} 0 &= \int_{B_1 \setminus B_a} \{ |\nabla v|^p - |v|^{p^*} - \lambda |v|^q \} \\ &\ge \int_{B_1 \setminus B_a} \{ |\nabla v|^p - |v|^{p^*} - \lambda C_1 |v|^p - \lambda C_2 |v|^{p^*} \} \end{aligned}$$

and

$$(1 + \lambda C_2) \int_{B_1 \setminus B_a} |v|^{p^*} \ge \left(1 - \frac{\lambda}{\lambda_1} C_1\right) \int_{B_1 \setminus B_a} |\nabla v|^p, \tag{1.7}$$

where $\lambda_1 = \lambda_1(B_1)$ is the Poincaré constant for B_1. From the Sobolev inequality and (1.7) we get

$$\int_{B_1 \setminus B_a} |v|^{p^*} \ge [C_\lambda S]^{\frac{n}{p}}, \quad C_\lambda = \frac{\lambda_1 - \lambda C_1}{\lambda_1 (1 + \lambda C_2)},$$

where S is the best constant for the Sobolev embedding $D^{1,p}(\mathbb{R}^n) \to L^{p^*}(\mathbb{R}^n)$; the space $D^{1,p}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ in the norm

$$||u||^p_{D^{1,p}} = \int_{\mathbb{R}^n} |\nabla u|^p \, dx.$$

Applying Hölder's inequality, we obtain from (1.6) that

$$|v(\tau)| \geq \left[\frac{n}{\omega_n(1-a^n)}\right]^{\frac{1}{p^*}} [C_\lambda S]^{\frac{n}{pp^*}}$$

and

$$|v'(a)| \geq \frac{n-p}{a(p-1)}\left(\frac{n}{\omega_n}\right)^{\frac{1}{p^*}} [C_\lambda S]^{\frac{n-p}{p^2}}. \tag{1.8}$$

Combining (1.4) and (1.8) we reach the conclusion of part a).

In order to show part b) we fix $\lambda > 0$ and suppose $v(r)$ has exactly $k-1$ nodes, $r_1, \cdots, r_{k-1}$, $a = r_0 < \cdots < r_{k-1} < r_k = 1$. Assume $\tau_j \in]r_j, r_{j+1}[$ such that $|v_j(\tau_j)| = \sup |v_j(r)|$ where

$$v_j(r) = \begin{cases} v(r) & \text{if } r \in]r_j, r_{j+1}[\text{ for } j = 0, 1, \cdots, k-1 \\ 0 & \text{otherwise.} \end{cases}$$

Then we obtain

$$F(v(\tau_{j-1})) - F(v(\tau_j)) = \int_{\tau_{j-1}}^{\tau_j} \frac{n-1}{r}|v'|^p \, dr > 0,$$

where $F(u) = \frac{1}{p^*}|u|^{p^*} + \frac{\lambda}{q}|u|^q$. Therefore,

$$|v_0(\tau_0)| > |v_1(\tau_1)| > \cdots > |v_{k-1}(\tau_{k-1})|. \tag{1.9}$$

Now, we can choose some j, $0 \leq j \leq k-1$, such that volumn of the support of v_j is less than $\frac{\omega_n}{nk}$. By the symmetrization method [5], we have

$$\int |\nabla v_j|^p \geq \lambda_1 k^{p/n} \int |v_j|^p.$$

Note that $\lambda_1(B_R) = \lambda_1(B_1)/R^p$. Using the estimate

$$\begin{aligned} 0 &= \int \{|\nabla v_j|^p - |v_j|^{p^*} - \lambda |v_j|^q\} \\ &\geq \left(1 - \frac{\lambda_1 C_1}{\lambda_1 k^{p/n}}\right) \int |\nabla v_j|^p - (1 + \lambda C_2) \int |v_j|^{p^*}, \end{aligned}$$

we obtain

$$\int |v_j|^{p^*} \geq [C_{k,\lambda} S]^{\frac{n}{p}}, \quad C_{k,\lambda} = \frac{\lambda_1 k^{p/n} - \lambda C_1}{\lambda_1 k^{p/n}(1+\lambda C_2)}.$$

Then we have

$$|v_j(\tau_j)| \geq \left(\frac{nk}{\omega_n}\right)^{\frac{1}{p^*}} [C_{k,\lambda} S]^{\frac{n}{pp^*}}$$

which combined with (1.6) and (1.9) implies

$$|v'(a)| \geq \frac{n-p}{a(p-1)} \left(\frac{nk}{\omega_n}\right)^{\frac{1}{p^*}} [C_{k,\lambda} S]^{\frac{n-p}{p^2}}. \tag{1.10}$$

From (1.4) and (1.10) we complete the proof of part b). □

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