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INTERSECTIONS OF MAXIMAL FACES IN THE CONVEX SET OF POSITIVE LINEAR MAPS BETWEEN MATRIX ALGEBRAS

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ABSTRACT. Let $\mathcal{P}_I$ be the convex compact set of all unital positive linear mpas between the $n \times n$ matrix algebra over the complex field. We find a necessary and sufficient condition for which two maximal faces of $\mathcal{P}_I$ intersect. In particular, we show that any pair of maximal faces of $\mathcal{P}_I$ has the nonempty intersection, whenever $n \geq 3$.

1. Introduction

Let M_n be the C^*-algebra of all $n \times n$ matrices over the complex field, and $\mathcal{P}$ the convex set of all positive linear maps between M_n, that is, which send positive semi-definite matrices into themselves. The structures of $\mathcal{P}$ are very complicated even in low dimensions, and several authors have tried to understand the structures of $\mathcal{P}$. For example, Størmer [**S**] found all extreme points of the convex set $\mathcal{P}_I$ of all unital positive linear maps for the case of $n = 2$, from which we know that every positive linear map between M_2 is decomposable. On the other hand, there are several examples of indecomposable maps which generate extreme rays if $n \geq 3$ [**CL, KK, O, R, W**].

In order to investigate the facial structures of a convex subset C of $\mathcal{P}$, the first author [**K**] have constructed a join homomorphism Φ_C from the complete lattice $\mathcal{F}(C)$ of all faces of the convex set C into the complete lattice $\mathcal{J}(\mathcal{V})$ of all join homomorphisms from $\mathcal{V}$ into itself, where $\mathcal{V}$ is the subspace lattice consisting of all subspaces of $\mathbb{C}^n$, or equivalently the

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lattice of all self-adjoint projections in $\mathbb{C}^n$. Using this machinery, it has been shown that every maximal face of $\mathcal{P}_I$ is of the form

$$F_I[\xi,\eta] = \{\phi \in \mathcal{P}_I : \phi([\xi])\eta = 0\},$$

where $[\xi]$ denotes the one-dimensional self-adjoint projection onto the subspace spanned by ξ. If we consider a unit vector $\xi \in \mathbb{C}^n$ as an $n \times 1$ matrix, $[\xi]$ is nothing but $\xi\xi^* \in M_n$. Furthermore, every two maximal faces are affine isomorphic to each other. In this note, we naturally look at intersections of maximal faces in order to investigate facial structures of $\mathcal{P}_I$. First of all, we find conditions for which two maximal faces have nonempty intersection. Especially, we show that any two maximal faces intersect each other, whenever $n \geq 3$. Note that the finite-dimensional compact convex set $\mathcal{P}_I$ has infinitely many maximal faces.

2. The case of $n \geq 3$

For a matrix $U \in M_n$, we denote by σ_U and τ_U the positive linear maps given by

$$\sigma_U : X \mapsto U^*XU, \quad \tau_U : X \mapsto U^*X^{\mathrm{tr}}U, \qquad X \in M_n,$$

respectively, where X^{tr} denotes the transpose of X. It is well-known that $\phi \in \mathcal{P}$ is completely positive (respectively completely copositive) if and only if ϕ is the convex combination of σ_U's (respectively τ_U's) [**C**]. When $n = 2$, every $\phi \in \mathcal{P}$ is the sum of a completely positive map and a completely copositive map, that is, every $\phi \in \mathcal{P}$ is decomposable as was mentioned in Introduction. Note that σ_U is unital if and only if U is a unitary.

LEMMA 2.1. *For a positive linear map $\phi \in \mathcal{P}_I$ given by*

$$\phi = \sum_{i=1}^{s} \sigma_{U_i} + \sum_{j=1}^{t} \tau_{V_j}, \tag{2.1}$$

the following are equivalent:

(i) $\phi \in F_I[\xi,\eta]$.

(ii) $\langle \bar{\xi}, U_i\eta\rangle = \langle \xi, V_j\eta\rangle = 0$ *for each* $i = 1,\ldots,s$ *and* $j = 1,\ldots,t$, *where* $\bar{\xi}$ *denotes the vector whose entries are complex conjugates of the corresponding entries of* ξ.

PROOF. We may assume that ξ and η are unit vectors. Then, we have $[\xi] = \xi\xi^*$ and $[\xi]^{\text{tr}} = \bar{\xi}\bar{\xi}^*$, and so

$$\begin{aligned}\phi \in F_I[\xi, \eta] &\iff \left(\sum_i U_i^* \bar{\xi}\bar{\xi}^* U_i + \sum_j V_j^* \xi\xi^* V_j\right)\eta = 0 \\ &\iff \left\langle \left(\sum_i U_i^* \bar{\xi}\bar{\xi}^* U_i + \sum_j V_j^* \xi\xi^* V_j\right)\eta, \eta \right\rangle = 0 \\ &\iff \sum_i |\bar{\xi}^* U_i \eta|^2 + \sum_j |\xi^* V_j \eta|^2 = 0.\end{aligned}$$

The required condition (ii) follows from the last formula. $\square$

THEOREM 2.2. *Assume that $n \geq 3$. Then we have*

$$F_I[\xi_1, \eta_1] \cap F_I[\xi_2, \eta_2] \neq \emptyset$$

for any pairs (ξ_1, η_1) and (ξ_2, η_2) of unit vectors.

PROOF. First, we assume that $[\eta_1] \neq [\eta_2]$. Put

$$\zeta = \eta_2 - \langle \eta_2, \eta_1 \rangle \eta_1,$$

and take unit vectors ζ_1 and ζ_2 which are orthogonal to the subspaces spanned by $\{\xi_1, \xi_2\}$ and $\{\zeta_1, \xi_2\}$, respectively. Since

$$\langle \eta_1, \zeta/\|\zeta\| \rangle = \langle \zeta_1, \zeta_2 \rangle = 0,$$

we may take a unitary $U \in M_n(\mathbb{C})$ such that

$$U\eta_1 = \zeta_1, \qquad U(\zeta/\|\zeta\|) = \zeta_2.$$

Then we have

$$U\eta_2 = U\zeta + \langle \eta_2, \eta_1 \rangle U\eta_1 = \|\zeta\|\zeta_2 + \langle \eta_2, \eta_1 \rangle \zeta_1,$$

and so, it follows that

$$\langle \xi_1, U\eta_1 \rangle = \langle \xi_1, \zeta_1 \rangle = 0,$$
$$\langle \xi_2, U\eta_2 \rangle = \|\zeta\| \langle \xi_2, \zeta_2 \rangle + \langle \eta_2, \eta_1 \rangle \langle \xi_2, \zeta_1 \rangle = 0.$$

By Lemma 2.1, we have $\sigma_U \in F_I[\xi_1, \eta_1] \cap F_I[\xi_2, \eta_2]$.

In case of $[\eta_1] = [\eta_2]$, we may assume that $\eta_1 = \eta_2$. Take a unit vector ζ orthogonal to ξ_1 and ξ_2, and a unitary U which sends η to ζ. Then we see that

$$\langle \xi_i, U\eta_i \rangle = \langle \xi_i, \zeta \rangle = 0, \qquad i = 1, 2.$$

Therefore, we have $\sigma_U \in F[\xi_1, \eta_1] \cap F[\xi_2, \eta_2]$. □

3. The case of $n = 2$

Now, we assume that $n = 2$. In this case, Theorem 2.2 is no longer true. To see this, we consider the case $[\eta_1] = [\eta_2]$. We may assume that $\eta_1 = \eta_2$, and put $\eta = \eta_1 = \eta_2$. If the map ϕ in (2.1) lies in $F_I[\xi_1, \eta] \cap F_I[\xi_2, \eta]$ then we have

$$\langle \xi_1, U_i\eta \rangle = \langle \xi_2, U_i\eta \rangle = \langle \xi_1, V_j\eta \rangle = \langle \xi_2, V_j\eta \rangle = 0,$$

for each $i = 1, \ldots, s$ and $j = 1, \ldots, t$. If $[\xi_1] \neq [\xi_2]$ then we also have $[\bar{\xi}_1] \neq [\bar{\xi}_2]$, and so we have

$$U_i\eta = V_j\eta = 0, \qquad i \in I, \ j \in J.$$

Therefore, it follows that

$$\phi(I)\eta = \sum_{i \in I} U_i^* U_i \eta + \sum_{j \in J} V_j^* V_j \eta = 0,$$

which is impossible since $\phi(I) = I$. This kind of restriction always occurs in the case of $n = 2$. We begin with the following simple lemma:

LEMMA 3.1. *Let α and β be complex numbers in the unit disc. Then the following are equivalent:*

(i) $|\alpha| \geq |\beta|$.

(ii) *There are vectors $x, y \in \mathbb{C}^s$ and $z, w \in \mathbb{C}^t$ for some s, t such that*

$$\|x\|^2 + \|z\|^2 = 1, \quad \|y\|^2 + \|w\|^2 = 1, \tag{3.1}$$
$$\alpha\langle x, y\rangle + \alpha\langle z, w\rangle = \beta.$$

PROOF. Assume that there are vectors x, y, z and w with the relation (3.1). Define $X, Y \in \mathbb{C}^{s+t}$ by

$$X = \begin{pmatrix} x \\ z \end{pmatrix}, \qquad Y = \begin{pmatrix} \bar{\alpha} y \\ \alpha w \end{pmatrix}$$

Then we have $\|X\| = 1$, $\|Y\| = |\alpha|$ and

$$\langle X, Y\rangle = \alpha\langle x, y\rangle + \bar{\alpha}\langle z, w\rangle = \beta,$$

from which the relation $|\alpha| \geq |\beta|$ follows. For the converse, we may assume that $\alpha \neq 0$. In this case, put

$$x = 1, \; z = 0, \; y = \frac{\bar{\beta}}{\bar{\alpha}}, \; w = \sqrt{1 - \frac{|\beta|^2}{|\alpha|^2}},$$

with $s = t = 1$. □

THEOREM 3.2. *Assume that $n = 2$. For unit vectors $\xi_1, \xi_2, \eta_1, \eta_2$ in $\mathbb{C}^2$, the following are equivalent:*

(i) $F_I[\xi_1, \eta_1] \cap F_I[\xi_2, \eta_2] \neq \emptyset$.

(ii) $|\langle\xi_1, \xi_2\rangle| \geq |\langle\eta_1, \eta_2\rangle|$.

PROOF. If $\phi \in F_I[\xi_1, \eta_1] \cap F_I[\xi_2, \eta_2]$ then ϕ is of the form in (2.1), because every positive linear map between M_2 is decomposable. Take a matrix $W \in M_2$ such that

$$We_1 = \eta_1, \qquad We_2 = \eta_2. \tag{3.2}$$

We also take unit vectors ζ_1 and ζ_2 orthogonal to ξ_1 and ξ_2, respectively. Then for each $k = 1, 2$ and $i = 1, 2, \ldots, s$, we have

$$\langle \xi_k, U_i W e_k \rangle = \langle \xi_k, U_i \eta_k \rangle = 0$$

by Lemma 2.1. Therefore, it follows that

$$U_i W e_1 = a_i \zeta_1, \qquad U_i W e_2 = b_i \zeta_2 \tag{3.3}$$

for some $a_i, b_i \in \mathbb{C}$. We also have

$$\langle \bar{\xi}_k, V_j W e_k \rangle = \langle \bar{\xi}_k, V_j \eta_k \rangle = 0,$$

for each $k = 1, 2$ and $j = 1, \ldots, t$. Since. $\langle \bar{\zeta}_k, \bar{\xi}_k \rangle = 0$ for $k = 1, 2$, we also have

$$V_j W e_1 = c_j \bar{\zeta}_1, \qquad V_j W e_2 = d_j \bar{\zeta}_2 \tag{3.4}$$

for some $c_j, d_j \in \mathbb{C}$. Since $\phi(I) = I$, we have

$$\begin{aligned} W^* W = \sigma_W(I) &= (\sigma_W \circ \phi)(I) \\ &= \sum_{i=1}^{s} W^* U_i^* U_i W + \sum_{j=1}^{t} W^* V_j^* V_j W. \end{aligned}$$

With (3.2), (3.3) and (3.4), this identity becomes

$$\begin{aligned} \begin{pmatrix} 1 & \langle \eta_2, \eta_1 \rangle \\ \langle \eta_1, \eta_2 \rangle & 1 \end{pmatrix} &= \sum_{i=1}^{s} \begin{pmatrix} |a_i|^2 & \bar{a}_i b_i \langle \zeta_2, \zeta_1 \rangle \\ a_i \bar{b}_i \langle \zeta_1, \zeta_2 \rangle & |b_i|^2 \end{pmatrix} \\ &\quad + \sum_{j=1}^{t} \begin{pmatrix} |c_j|^2 & \bar{c}_j d_j \langle \zeta_1, \zeta_2 \rangle \\ c_j \bar{d}_j \langle \zeta_2, \zeta_1 \rangle & |d_j|^2 \end{pmatrix} \end{aligned}$$

If we put

$$x = \begin{pmatrix} a_1 \\ \vdots \\ a_s \end{pmatrix}, \quad y = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix}, \quad z = \begin{pmatrix} c_1 \\ \vdots \\ c_t \end{pmatrix}, \quad w = \begin{pmatrix} d_1 \\ \vdots \\ d_t \end{pmatrix}, \tag{3.5}$$

then the above relation gives us

$$\begin{aligned} &\|x\|^2 + \|z\|^2 = 1, \qquad \|y\|^2 + \|w\|^2 = 1, \\ &\langle x, y\rangle\langle\zeta_1, \zeta_2\rangle + \langle z, w\rangle\langle\zeta_2, \zeta_1\rangle = \langle\eta_1, \eta_2\rangle. \end{aligned} \tag{3.6}$$

By Lemma 3.1, we have

$$|\langle\eta_1, \eta_2\rangle| \le |\langle\zeta_1, \zeta_2\rangle| = |\langle\xi_1, \xi_2\rangle|$$

as was required.

For the converse, we first consider the case $|\langle\eta_1, \eta_2\rangle| = 1$. In this case, we also have $|\langle\xi_1, \xi_2\rangle| = 1$. Therefore, it follows that $[\xi_1] = [\xi_2]$ and $[\eta_1] = [\eta_2]$, from which we have

$$F_I[\xi_1, \eta_1] = F_I[\xi_2, \eta_2].$$

Now, assume that $|\langle\eta_1, \eta_2\rangle| < 1$. Taking unit vectors ζ_1 and ζ_2 orthogonal to ξ_1 and ξ_2, respectively, we have

$$|\langle\eta_1, \eta_2\rangle| \le |\langle\zeta_1, \zeta_2\rangle|.$$

By Lemma 3.1, there are vectors $x, y \in \mathbb{C}^s$ and $z, w \in \mathbb{C}^t$ satisfying the relation (3.6). We denote the entries of these vectors as in (3.5). Because $\{\eta_1, \eta_2\}$ is linearly independent, we may take matrices $W, U_1, \dots, U_s$ and $V_1, \dots, V_t$ satisfying the relations (3.2), (3.3) and (3.4). Then it is clear that the map ϕ defined by (2.1) lies in $F_I[\xi_1, \eta_1] \cap F_I[\xi_2, \eta_2]$. □

In the case of $[\eta_1] = [\eta_2]$, we see that $F_I[\xi_1, \eta_1] \cap F_I[\xi_2, \eta_2]$ is nonempty if and only if $[\xi_1] = [\xi_2]$ if and only if the two sets coincide as in the proof of Theorem 3.2. On the other hand, if $[\xi_1] = [\xi_2]$ then two sets have the nonempty intersection by the theorem. Take ϕ_k ($k = 1, 2$) in this intersection. If $[\eta_1] \ne [\eta_2]$ then $\phi_k([\xi_1]) = 0$ and so

$$\phi_k(I - [\xi_1]) = I - \phi_k([\xi_1]) = I, \qquad k = 1, 2.$$

Take unitary U such that $Ue_1 = \xi_1$. Then the map $\psi_k = \phi_k \circ \sigma_U$ sends e_{11} and e_{22} to 0 and I, respectively, where $\{e_{ij}\}$ is the usual matrix units. Form this, it is easy to see that $\psi_k(e_{ij}) = \psi_k(e_{ji}) = 0$, and so $\psi_1 = \psi_2$. Therefore, the set $F_I[\xi_1, \eta_1] \cap F_I[\xi_2, \eta_2]$ consists of a single map, if $[\xi_1] = [\xi_2]$ and $[\eta_1] \ne [\eta_2]$.

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