Comm. Korean Math. Soc. 10 (1995), No. 4, pp. 907–916

ON THE UNICELLULARITY OF AN OPERATOR

JOO HO KANG AND YOUNG SOO JO

ABSTRACT. The unilateral weighted shift operator W_r with the weighte sequence $\{r^n\}_{n=0}^{\infty}$ is unicellular if $0 < r < 1$. In general, $A + B$ is not unicellular even if A and B are unicellular. We will prove that $W_r + W_r^2$ is unicellular if $0 < r < 1$.

1. Introduction

The problem of invariant subspaces is one of the most important, most difficult, and most exasperating problems of operator theory. One of the questions about invariant subspaces is the following : is there an operator whose lattice of invariant subspaces is isomorphic to the positive integers? In other words, is there an operator for which there is an one-to-one and order-preserving correspondence $n \longmapsto M_n$, for each $n = 0, 1, 2, \cdots, \infty$, between the indicated integers(including ∞) and all invariant subspaces? We have such well-known operators ; Donoghue weighted shift operator, Volterra operator, Discrete Volterra operator, etc. And there are many ways to solve the problem. We have investigated and found a sufficient and necessary condition which a strictly lower triangular operator can be unicellular and showed the unicellularity of the Donoghue weighted shift operator under a certain condition by the way in [7].

In this paper we want to show that an operator $W_r + W_r^2$ is unicellular under a certain condition related r, where $W_r + W_r^2$ is a strictly lower triangular matrix whose the first off diagonal is $\{r^n\}_{n=0}^{\infty}$ and the second off diagonal is $\{r^{2n-1}\}_{n=1}^{\infty}$.

Received May 24, 1995.
1991 AMS Subject Classification: 47A15, 47B37.
Key words: invariant subspace, unicellular, unilateral weighted shift.
This paper is partially supported by KOSEF 94-1400-02-01-3 and TGRC-KOSEF 1995.

We first introduce some definitions and given theorems. Let $\mathcal{H}$ be a Hilbert space and A an operator on $\mathcal{H}$. Let M denote a subspace of $\mathcal{H}$. M is invariant under A means that $Ax \in M$ for all $x \in M$. The collection of all subspaces of $\mathcal{H}$ invariant under A is denoted by Lat$A$. An operator A is unicellular if the collection Lat$A$ is totally ordered by inclusion. If K is a subset of $\mathcal{H}$, the span of K is the smallest subspace containing K and denoted by $spanK$. If $x \in \mathcal{H}$ then $span\{x, Ax, A^2x, \cdots\}$ is invariant under A. The vector x is cyclic for A if $span\{x, Ax, A^2x, \cdots\}=\mathcal{H}$ and M is a cyclic subspace for A if $span\{x, Ax, A^2x, \cdots\} = M$ for some $x \in \mathcal{H}$.

Let A be a bounded operator with $\| A \| < 1$ on ℓ^2, and let $\{e_0, e_1, e_2, \cdots\}$ denote the standard basis for ℓ^2. Let x be a column vector in ℓ^2. Then $A^n x$ is a column vector in ℓ^2 for each $n = 1, 2, \cdots$. Then we have an infinite matrix $[x, Ax, A^2x, \cdots]^t$ which will be denoted by $S_x(A)$. The matrix $S_x(A)$ is a bounded linear transformation on ℓ^2. Let A be an bounded operator with $\| A \| < 1$ on ℓ^2 represented by a strictly lower triangular matrix. Let M_n be the subspace $span\{e_n, e_{n+1}, e_{n+2}, \cdots\}$ for each $n = 0, 1, 2, \cdots$. Then each M_n is invariant under A, and $\{M_n \mid n = 0, 1, 2, \cdots\}$ is totally ordered by inclusion :

$$\ell^2 = M_0 \supset M_1 \supset M_2 \supset \cdots .$$

Hence A is unicellular if its only invariant subspaces are $\{0\}$ and M_n, $n = 0, 1, 2, \cdots$, i.e. the collection Lat$A$ of all subspaces of ℓ^2 which are invariant under A is $\{\{0\}, M_n \mid n = 0, 1, 2, \cdots\}$. Let M be a subspace of ℓ^2 and $M^* = \{ \sum_{n=0}^{\infty} \bar{c}_n e_n \ : \ \sum_{n=0}^{\infty} c_n e_n \in M \}$. If we let

$$\underline{x}_N = (\underbrace{0, \cdots, 0}_{N-terms}, 1, x_{N+1}, \cdots) \in \ell^2, \quad M_{\underline{x}_N} = span\{\underline{x}_N, A\underline{x}_N, A^2\underline{x}_N, \cdots\}$$

then $M_{\underline{x}_N}{}^{\perp} = (KerS_{\underline{x}_N}(A))^*$ and always $M_N{}^{\perp} \subset (KerS_{\underline{x}_N}(A))^*$.

LEMMA 1 [7]. *Let A be a strictly lower triangular operator on ℓ^2. Then $(U^{*N}AU^N)^n U^{*N} = U^{*N}A^n P_N$ for every $n, N = 0, 1, 2, \cdots$, where U is the unilateral shift on ℓ^2 and P_N the orthogonal projection on M_N.*

LEMMA 2 [7]. *Let A be a strictly lower triangular operator with $\|A\| < 1$ on ℓ^2, N a non-negative integer and let $\underline{x}_N = (\underbrace{0, \cdots, 0}_{N-terms}, 1,$*

$x_{N+1}, \cdots)^t \in M_N$. *$M_N$ is a cyclic subspace for A, i.e. $M_N = M_{\underline{x}_N}$, if and only if $S_{U^{*N}\underline{x}_N}(U^{*N}AU^N)$ is one-to-one.*

THEOREM 3 [7]. *Let A be a strictly lower triangular operator with $\| A \| < 1$ and U the unilateral shift on ℓ^2. Then A is unicellular if and only if for any $x = (1, x_1, \cdots)^t \in \ell^2$, $S_x(U^{*N}AU^N)$ is one-to-one for every $N = 0, 1, 2, \cdots$.*

We need some properties of strictly upper triangular matrices in order to determine whether they are one-to-one . The following theorem leads to results on unicellularity.

THEOREM 4 [7]. *Let T and S be bounded operators on a Hilbert space $\mathcal{H}$ represented by upper triangular matrices with respect to a fixed orthonormal basis. Assume that all diagonal entries of T are non-vanishing, and that all diagonal entries of S are 0. If T is invertible and S is compact, then $T + S$ is one-to-one.*

2. The unicellularity of $W_r + W_r^2$

In this section, we will study an extension of the unilateral weighted shift operators and apply the previous techniques.

PROPOSITION 5. *If A is compact and has a strictly lower triangular representation with respect to some basis, then A is quasi-nilpotent.*

PROOF. Since A is compact, $\sigma(A) = \{0\} \cup \pi_0(A)$. So, it is enough to show that $\pi_0(A) \cap (C - \{0\}) = \emptyset$. Else, there exists $\alpha \neq 0$ and $x \neq 0$ in ℓ^2 such that $Ax = \alpha x$. By hypothesis, $A = \{A_{ij}\}_{i,j=0}^{\infty}$ with respect to a fixed orthonomal basis and $A_{ij} = 0$ if $j \geq i$. But this entails $0 = \alpha x_0, A_{10}x_0 = \alpha x_1, A_{20}x_0 + a_{21}x_1 = \alpha x_2, \cdots, A_{n0}x_0 + A_{n1}x_1 + \cdots + A_{nn-1}x_{n-1} = \alpha x_n, \cdots$. Thus $0 = x_0 = x_1 = \cdots$. So $x = 0$ which is a contradiction.

Let W_r^* be the Donoghue weighted shift operator with the weight sequence $\{r^n\}_{n=0}^{\infty}$. Then

$$W_r = \begin{bmatrix} 0 & 1 & 0 & & & \\ & 0 & r & 0 & & 0 \\ & & 0 & r^2 & 0 & \\ & 0 & & \ddots & \ddots & \ddots \end{bmatrix}.$$

In this case, if $r < 1$, then $\{r^n\}_{n=0}$ is monotone and belongs to ℓ^2. Therefore, W_r^* is unicellular [7].

We consider the following operator, when $0 < r < 1$.

$$W_r + W_r^2 = \begin{bmatrix} 0 & 1 & r & 0 & & & \\ & 0 & r & r^3 & 0 & & 0 \\ & & 0 & r^2 & r^5 & 0 & \\ & 0 & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Now $W_r + W_r^2 = W_r(I + W_r)$. So,

$$\begin{aligned} &(W_r + W_r^2)^n \\ =&W_r^n(I + W_r)^n \\ =&W_r^n(I + {}_nC_1 W_r + {}_nC_2 W_r^2 + \cdots + {}_nC_{n-1} W_r^{n-1} + W_r^n) \\ =&W_r^n + {}_nC_1 W_r^{n+1} + {}_nC_2 W_r^{n+2} + \cdots + {}_nC_{n-1} W_r^{2n-1} + W_r^{2n} \end{aligned}$$

First we find W_r^n. $\quad W_r e_j = \begin{cases} 0 & j = 0\,, \\ r^{j-1} e_{j-1} & j \geq 1 \end{cases}.$

Then

$$W_r^n e_j = \begin{cases} 0 & j < n, \\ r^{j-1} \cdots r^{j-n} e_{j-n} = r^{nj-n(n+1)/2} e_{j-n} & j \geq n \end{cases}.$$

So,

$$(W_r^n)_{kj} = < W_r^n e_j, e_k > = \begin{cases} r^{nj-n(n+1)/2} = r^{n(n+2k-1)/2} & k = j-n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$W_r^n = \begin{bmatrix} \overbrace{0\cdots 0}^{n-terms} & r^{n(n-1)/2} & 0 & \cdots & \\ 0\cdots 0 & 0 & r^{n(n+1)/2} & 0 & \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$[(W_r + W_r^2)^n]_{kj} = (W_r^n + {}_nC_1 W_r^{n+1} + \cdots + {}_nC_{n-1} W_r^{2n-1} + W_r^{2n})_{kj}$$

$$= \begin{cases} {}_nC_{j-k-n}(W_r^{j-k})_{k,j} = {}_nC_{j-k-n} r^{\frac{1}{2}(j-k)(j+k-1)} & n \leq j-k \leq 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$[(W_r + W_r^2)^{*^n}]_{jk} = [(W_r + W_r^2)^n]_{kj} = \begin{cases} {}_nC_{j-k-n} r^{\frac{1}{2}(j-k)(j+k-1)} & n \leq j-k \leq 2n \\ 0 & \text{otherwise.} \end{cases}$$

$$(W_r + W_r^2)^{*^n} = \begin{bmatrix} 0 & & \\ \vdots & & \\ 0 & & \\ r^{n(n-1)/2} & 0 & \\ {}_nC_1 r^{(n+1)n/2} & r^{n(n+1)/2} & 0 \\ \vdots & \vdots & \ddots \\ r^{(2n-1)2n/2} & {}_nC_{n-1} r^{(2n-1)2n/2} & \cdots \\ 0 & r^{2n(2n+1)/2} & \cdots \\ & 0 & \\ & & \ddots \end{bmatrix}$$

Let $x = (1, x_1, \cdots)^t \in \ell^2$. Then

$$[(W_r + W_r^2)^{*^n} x]_{n+m} = \sum_{k=0}^{m} [(W_r + W_r^2)^{*^n}]_{n+m,k} x_k = \sum_{k=0}^{m} {}_nC_{m-k} r^{(n+m-k)(n+m+k-1)/2} x_k$$

where $m = 0, 1, 2, \cdots$. But, if $m - k < 0$ or $m - k > n$, we will set ${}_nC_{m-k} = 0$. So,

$$[(W_r + W_r^2)^{*^n} x]_{n+m} \tag{1}$$
$$= \begin{cases} \sum_{k=0}^{m} {}_nC_{m-k} r^{(n+m-k)(n+m+k-1)/2} x_k & 0 \le m \le n, \\ \sum_{k=0}^{n} {}_nC_{n-k} r^{(n+m-k)(n+m+k-1)/2} x_{m-n+k} & n < m \end{cases}$$

$$(W_r + W_r^2)^{*^n} x = \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} n - terms \\ r^{n(n-1)/2} \\ {}_nC_1 r^{(n+1)n/2} + r^{n(n+1)/2} x_1 \\ \vdots \\ r^{2n(2n-1)/2} + {}_nC_{n-1} r^{(2n-1)2n/2} x_1 \\ + \cdots + r^{n(n+2n-1)/2} x_n \\ r^{2n(2n+1)/2} x_1 + {}_nC_{n-1} r^{(2n-1)(2n+2)/2} x_2 \\ + \cdots + r^{n(n+2n+1)/2} x_{n-1} \\ \vdots \end{bmatrix}$$

Now we consider conditions sufficient to ensure that $S_x[(W_r + W_r^2)^*]$ is one-to-one. $S_x[(W_r + W_r^2)^*] = [x, (W_r + W_r^2)^* x, (W_r + W_r^2)^{*^2} x, \cdots]^t$, i.e. $S_x[(W_r + W_r^2)^*]_{n,n+m} = [(W_r + W_r^2)^{*^n} x]_{n+m}$, for $n \ge 0$ and $m \ge 0$.

$$S_x((W_r + W_r^2)^*) = \begin{bmatrix} 1 & x_1 & x_2 & x_3 & \cdots \\ 0 & 1 & r + rx_1 & r^3 x_1 + r^2 x_2 & \cdots \\ 0 & 0 & r & 2r^3 + r^3 x_1 & \cdots \\ 0 & 0 & 0 & r^3 & \cdots \\ \vdots & & & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ & & r & \\ & 0 & r^3 & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_2 & x_3 & \cdots \\ 0 & 1 & r + rx_1 & r^3 x_1 + r^2 x_2 & \cdots \\ 0 & 0 & 1 & 2r^2 + r^2 x_1 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \end{bmatrix}$$

$$= DA.$$

Then,

$$\begin{aligned} \mid A_{n,n+m} \mid &= \mid [(W_r + W_r^2)^{*^n} x]n + m / [(W_r + W_r^2)^{*^n}]_{n,0} \mid \\ &\le \sum_{k=0}^{m} {}_nC_{m-k} r^{(n+m-k)(n+m-1+k)/2 - n(n-1)/2} \mid x_k \mid \\ &= \sum_{k=0}^{m} {}_nC_{m-k} r^{nm} r^{(m-k)(m+k-1)/2} \mid x_k \mid \\ &\le \sum_{k=0}^{m} {}_nC_{m-k} r^{nm} r^{(m-1)/2} \mid x_k \mid \\ &\le \sum_{k=0}^{m} {}_nC_{m-k} r^{nm} \mid x_k \mid \quad \text{for all } m \ge 1 \end{aligned} \tag{2}$$

$$| A_{n,n+m} |^2 \leq \sum_{k=0}^{m} ({}_nC_{m-k})^2 r^{2nm} \sum_{k=0}^{m} | x_k |^2$$
$$\leq \mathrm{M} \sum_{k=0}^{m} ({}_nC_{m-k})^2 r^{2nm} \text{ for all } m \geq 1$$

where $\mathrm{M} = \sum_n | x_n |^2 < \infty$.

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} | A_{n,n+m} |^2 \leq \mathrm{M} \, [\sum_{n=1}^{\infty} \sum_{m=1}^{n} r^{2nm} \sum_{k=0}^{m} ({}_nC_{m-k})^2 + \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} r^{2nm} \sum_{k=0}^{n} ({}_nC_{n-k})^2].$$

Let
$$I = \sum_{n=1}^{\infty} \sum_{m=1}^{n} r^{2nm} \sum_{k=0}^{m} ({}_nC_{m-k})^2 \text{ and}$$
$$II = \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} r^{2nm} \sum_{k=0}^{n} ({}_nC_{n-k})^2.$$

It is sufficient to show that I and II are finite.

$$({}_nC_{m-k})^2 = [n(n-1)\cdots(n-(m-k)+1)]^2/[(m-k)!]^2$$
$$\leq n^{2(m-k)} \leq n^{2m}.$$

So, $\sum_{k=0}^{m} ({}_nC_{m-k})^2 \leq (m+1)n^{2m}$. Since $\sum_{k=0}^{n} {}_nC_{n-k} = 2^n$, $\sum_{k=0}^{n} ({}_nC_{n-k})^2 \leq 2^{2n}$. Hence $\sum_{k=0}^{m} ({}_nC_{m-k})^2 \leq min\{2^{2n}, (m+1)n^{2m}\}$ for $m \leq n$.

$$II = \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} r^{2nm} \sum_{k=0}^{n} ({}_nC_{n-k})^2$$
$$\leq \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} r^{2nm} 2^{2n}$$
$$= \sum_{n=1}^{\infty} 2^{2n} r^{2n(n+1)}/(1 - r^{2n})$$
$$\leq \sum_{n=1}^{\infty} (4r^{2(n+1)})^n/(1 - r).$$

Choose N_0 such that $4r^{2(n+1)} < \frac{1}{2}$ for all $n \geq N_0$. Then

$$II \leq \sum_{n=1}^{N_0}(4r^{2(n+1)})^n/(1-r) + \sum_{n=N_0+1}^{\infty}(\frac{1}{2})^n/(1-r)$$
$$= \sum_{n=1}^{N_0}(4r^{2(n+1)})^n/(1-r) + (\frac{1}{2})^{N_0}/(1-r) < \infty$$

Choose N_1 such that $r^n(m+1)^{1/m}n^2 < 1$ for all $n > N_1$ and $1 \leq m \leq n$. Then for all $n \geq N_1$, $r^{nm}(m+1)n^{2m} < 1$.
Then,

$$I \leq \sum_{n=1}^{N_1}\sum_{m=1}^{n} r^{2nm}\sum_{k=0}^{m}({}_nC_{m-k})^2 + \sum_{n=N_1+1}^{\infty}\sum_{m=1}^{n} r^{nm}(r^n(m+1)^{1/m}n^2)^m$$
$$\leq \sum_{n=1}^{N_1}\sum_{m=1}^{n} r^{2nm}\sum_{k=0}^{m}({}_nC_{m-k})^2 + \sum_{n=N_1+1}^{\infty}\sum_{m=1}^{n} r^{nm} \text{ and}$$

$$\sum_{n=N_1+1}^{\infty}\sum_{m=1}^{n} r^{nm} \leq r^{N_1+1}/(1-r)^2 < \infty$$

So, A is Hilbert-Schmidt. Therefore, $S_x((W_r + W_r^2)^*)$ is one-to-one by Theorem 4, and $(W_r + W_r^2)^*$ is unicellular. Hence we have the following Theorem.

THEOREM 6. *If $0 < r < 1$, then $(W_r + W_r^2)^*$ is unicellular.*

References

1. R. Beals, *Topics in operator theory*, The university of Chicago Press, Chicago and London (1971).
2. J. B. Conway, *A course in funtional analysis*, Springer-Verlag, N. Y. (1985).
3. J. B. Conway, *Subnormal Operators*, Pitman Advanced Pub. Program (1981).
4. I. Gohberg and S. Goldberg, *Basic operator theory* Birkhauser, Boston (1981).
5. P. R. Halmos, *A Hilbert space problem book(second edition)*, Springer-Verlag, N. Y. (1982).

6. K. J. Harrison, *On the unicellularity of weighted shifts*, J. Austral. Math. Soc. 12 (1982).
7. J. H. Kang, *On the unicellularity of triangular operators*, Houston J. of Math. Vol.19 No.1 (1993).
8. H. Rajavi and P. Rosenthal, *Invariant subspaces*, Sringer-Verlag, N. Y. (1970).
9. A. L. Shields, *Weighted shift operator and analytic function theory*, A. M. S., Topics in Operator Theory, 19-128 (1970).
10. M. Reed and B. Simon, *Functional analysis I*, Acdemic Press, N. Y. (1980).
11. J. Weidman, *Linear operators in Hilbert spaces*, Springer-Verlag, N. Y. (1980).
12. J. Wermer, *Banach Algebras and several complex variables*, Springer-Verlag, N. Y. (1976).

Joo Ho Kang
Dept. of Math.
Taegu Univ.
Taegu, Korea

Young Soo Jo
Dept. of Math.
Keimyung Univ.
Taegu, Korea