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EXTREME SPIRALLIKE PRODUCTS

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ABSTRACT. Let $S_p(\alpha)$ denote the class of the Spirallike functions of order α, $0 < |\alpha| < \frac{\pi}{2}$. Let Π_N denote the subset of $S_p(\alpha)$ consisting of all products $z\Pi_{j=1}^{N}(1-u_j z)^{-mt_j}$ where $m = 1 + e^{-2i\alpha}, |u_j| = 1,\ t_j > 0$ for $j = 1, \cdots, N$ and $\sum_{j=1}^{N} t_j = 1$. In this paper we prove that extreme points of $S_p(\alpha)$ may be found which lie in Π_N for some $N \geq 2$. We are let to conjecture that all exreme points of $S_p(\alpha)$ lie in Π_N for some $N \geq 1$ and that every such function is an extreme point.

1. Introduction

Let $S_p(\alpha)$ denote the class of the Spirallike functions of order α, $0 < |\alpha| < \frac{\pi}{2}$. These are the functions $f(z)$ analytic on the open unit disc $\triangle$ and satisfying $Re\frac{e^{i\alpha}zf'(z)}{f(z)} > 0$, with $f(0) = 0$ and $f'(0) = 1$.

Writing $m = 1 + e^{-2i\alpha}$, let Π_N denote the subset of $S_p(\alpha)$ consisting of all products $z\Pi_{j=1}^{N}(1-u_j z)^{-mt_j}$ where the N points u_j are distinct with $|u_j| = 1,\ t_j > 0$ for $j = 1, \cdots, N$ and $\sum_{j=1}^{N} t_j = 1$.

It is known that the functions in Π_1 are extreme points of $S_p(\alpha)$. It was conjectured by MacGregor [3] that these were the only ones. Pearce [5] showed that there are extreme points of $S_p(\alpha)$ not lying in Π_1. We prove that in fact extreme points of $S_p(\alpha)$ may be found which lie in Π_N for some $N \geq 2$. We are let to conjecture that all extreme points of $S_p(\alpha)$ lie in Π_N for some $N \geq 1$ and that every such function is an extreme point.

We use a number of standard results from [2] and [7]. The topology is that of uniform convergence on compact subsets of $\triangle$.

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2. Linear functionals on $S_p(\alpha)$

Any continuous complex linear functional on the span of the normalized univalent functions S may be written in the form
$J(f) = \sum_{j=1}^{\infty} \frac{f^{(j+1)}(0)}{(j+1)!}$ where $\limsup_{n\to\infty} |b_n|^{\frac{1}{n}} < 1$.

Pearce separated the Π_2 function $p(z) = z(1-z^2)^{-\frac{m}{2}}$ from the functions Π_1 using the continuous linear functional

$$\operatorname{Re} J_0(f) = \operatorname{Re} \sum_{j=0}^{\infty} (-1)^j x^j \binom{-m}{j}^{-1} \frac{f^{(j+1)}(0)}{(j-1)!}$$

where $0 < x < 1$.

We show that $\operatorname{Re} J_0$ may be replaced by a suitable finite-length approximation L.

PROPOSITION 1. *There is a continuous linear functional L on S of the form $L(f) = Re \sum_{j=1}^{N} b_j \frac{f^{(j+1)}(0)}{(j+1)!}$ which satisfies $L(p) < -\frac{1}{2} < 0 \leq$ $inf\{L(f) : f \in \Pi_1\}$ where $p(z) = z(1-z^2)^{-\frac{m}{2}}$.*

PROOF. Let J_0 be as above. For all functions $f(z) = z(1-uz)^{-m}$ in Π_1, where $|u| = 1$,

$$J_0(f) = \sum_{j=0}^{\infty} (-1)^j x^j \binom{-m}{j}^{-1} (-1)^j u^j \binom{-m}{j} = \sum_{j=0}^{\infty} x^j u^j = \frac{1}{1-xu}.$$

Since $(1-xu)^{-1}$ lies on the circle with diameter $\left(\frac{1}{1-x}, \frac{1}{1-x}\right)$, we have $\operatorname{Re} J_0(f) > \frac{1}{2}$. For the Π_2 function $p(z) = z(1-z^2)^{-\frac{m}{2}}$,

$$\begin{aligned} J_0(p) &= \sum_{j=0}^{\infty} (-1)^{2j} x^{2j} \binom{-m}{2j}^{-1} (-1)^j \binom{-\frac{m}{2}}{j} \\ &= \sum_{j=0}^{\infty} \frac{(\frac{1}{2})_j (1)_j}{(\frac{m+1}{2})_j j!} x^{2j} = {}_2F_1\left(\frac{1}{2}, 1;\ \frac{m+1}{2};\ x^2\right). \end{aligned}$$

Here ${}_2F_1$ denotes the hypergeometric function
${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$ where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for $n \geq 1$ and $(a)_0 = 1$.

Euler's identity asserts that
${}_2F_1(a,b;c;z) = (1-z)^{(c-a-b)}{}_2F_1(c-a,c-b;c;z)$ when $c \neq 0,-1,-2,\cdots$

Gauss's theorem is that
$\lim_{y\to 1^-} {}_2F_1(a,b;c;y) = {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \neq 0$ when $\mathrm{Re}(c-a-b) > 0$.

Since $\frac{1}{2}(m+1)$ is not an integer and $\mathrm{Re}\left(\frac{1}{2}(m+1) - \frac{1}{2}(m-1) - \frac{1}{2}m\right) = \sin^2\alpha > 0$

$$\begin{aligned} & J_0(f) \\ &= (1-x^2)^{\frac{m}{2}-1}{}_2F_1\left(\frac{m-1}{2}, \frac{m}{2}; \frac{m+1}{2}; x^2\right) \\ &= \pi^{-\frac{1}{2}}(1-x^2)^{-\sin^2\alpha} e^{-i\log(1-x^2)\frac{1}{2}\sin 2\alpha}\left(\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(1-\frac{m}{2}\right) + o(1)\right) \\ &= R(x)e^{i\phi(x)}(A + o(1)) \quad \text{as } x \to 1^-. \end{aligned}$$

Now as $x \to 1^-, R(x) \to \infty$ and $|\varphi(x)| \to \infty$, so x may be chosen to be a fixed value with $\mathrm{Re}\, J_0(p) < -1 < \frac{1}{2} \leq \inf\{\mathrm{Re}\, J_0(f) : f \in \Pi_1\}$.

Finally, choose N such that $\left|\sum_{j=N+1}^{\infty} \frac{(\frac{1}{2})_j(1)_j}{(\frac{m+1}{2})_j j!} x^{2j}\right| < \frac{1}{2}$ and $x^{N+1} < \frac{1}{2}(1-x)$.

Then the linear functional $L(f) = \mathrm{Re}\sum_{j=0}^{N} \frac{(\frac{1}{2})_j(1)_j}{(\frac{m+1}{2})_j j!} x^{2j}$ satisfies $L(p) < -\frac{1}{2} \leq 0 \leq \inf\{L(f) : f \in \Pi_1\}$. $\square$

We see that the closed hyperplane $L^{-1}(\{-\frac{1}{2}\})$ separates the function p from the set Π_1, also showing that p is not contained in the closed convex hull of Π_1.

3. Extremal functions for $L(f)$

Pinchuk [6] showed that for $L(f) = \mathrm{Re}\sum_{j=0}^{N} b_j \frac{f^{(j+1)}(\zeta)}{(j+1)!}$, $\zeta \neq 0$, L is maximised on $\mathrm{Sp}(\alpha)$ at a point of Π_n for some n with $1 \leq n \leq N+1$ and gave a special case of the following proposition.

PROPOSITION 2. *The linear functional* $L(f) = Re\sum_{j=0}^{N} b_j \frac{f^{(j+1)}(0)}{(j+1)!}$ *is maximised on* $Sp(\alpha)$ *only at points of* Π_n *with* $1 \leq n \leq N$.

PROOF. Every function f in $\mathrm{Sp}(\alpha)$ may be represented by

$$f(z) = z \exp\left(-m \int_{-\pi}^{\pi} \log(1 - ze^{it}) d\psi(t)\right)$$

where $\psi(t)$ is an increasing function on $[-\pi, \pi]$ with $\psi(-\pi) = 0$, $\psi(\pi) = 1$ and $m = 1 + e^{-2i\alpha}$.

Golusin's variational principle [1] states that for each pair t_1, t_2 with $-\pi \leq t_1 \leq t_2 \leq \pi$, there exists a constant C independent of z and t such that for all real λ in an open interval containing 0 the function

$$\begin{aligned} f_*(z) &= f(z) \exp\left(-m\lambda \int_{t_1}^{t_2} \frac{iz}{e^{it} - z} |\psi(t) - C| dt\right) \\ &= f(z) - \lambda m \int_{t_1}^{t_2} \frac{izf(z)}{e^{it} - z} |\psi(t) - C| dt + O(\lambda^2) \end{aligned}$$

lies in $\mathrm{Sp}(\alpha)$.

Applying L to f_* yields

$$\begin{aligned} L(f_*) &= L(f) - L\left(\lambda m \int_{t_1}^{t_2} \frac{izf(z)}{e^{it} - z} |\psi(t) - C| dt\right) + O(\lambda^2) \\ &= L(f) - \lambda \int_{t_1}^{t_2} \mathrm{Re}\, m \sum_{j=1}^{N} \frac{b_j}{(j+1)!} \left(\frac{d^{j+1}}{dz^{j+1}} \frac{izf(z)}{e^{it} - z}\right)_{z=0} \cdot |\psi(t) - C| dt \\ &\qquad + O(\lambda^2) \\ &= L(f) - \lambda \int_{t_1}^{t_2} Q(t) |\psi(t) - C| dt + O(\lambda^2) \end{aligned}$$

where $Q(t)$ is of the form $\mathrm{Re} \sum_{j=1}^{N} A_j e^{-ijt} = e^{-iNt} \sum_{j=0}^{2N} B_j e^{ijt}$.

Since L attains its maximum at f, the coefficient of λ vanishes. Now $Q(t)$ is continuous and has at most $2N$ zeros. So $\psi(t)$ is constant on any interval $[t_1, t_2]$ where Q does not change sign, and must be a step function with at most $2N$ jump points.

We take the opportunity to introduce more general one-sided version of Golusin's variation f_{**}. Provided $\psi(t)$ has at least two jump points then for each jump point t_j and sufficiently small $\delta > 0$ and $\lambda > 0$, the functions $\phi^{-}(t) = \psi(t) + \lambda\mathcal{X}_{[t_j-\delta,t_j]}(t)$ and $\phi^{+}(t) = \psi(t) - \lambda\mathcal{X}_{[t_j,t_j+\delta]}(t)$ are increasing and determine elements of $\mathrm{Sp}(\alpha)$ given by

$$f_{**}^{-}(z) = f(z) - \lambda m f(z)\left(\log(1 - ze^{-it_j}) - \log(1 - ze^{-i(t_j-\delta)})\right)$$

and

$$f_{**}^{+}(z) = f(z) + \lambda m f(z)\left(\log(1 - ze^{-i(t_j+\delta)}) - \log(1 - ze^{-it_j})\right).$$

Applying L we obtain $L(f_{**}^{-}) = L(f) - \lambda(R(t_j) - R(t_j - \delta))$ and $L(f_{**}^{+} = L(f) + \lambda(R(t_j + \delta) - R(t_j))$ where

$$R(t) = \operatorname{Re} m \sum_{j=1}^{N} \frac{b_j}{(j+1)!}\left(\frac{d^{j+1}}{dz^{j+1}} f(z)\log(1 - ze^{-it})\right)_{z=0}$$

and $R'(t) = Q(t)$.

Now since L attains its maximum at f, we have $R(t_j - \delta) \le R(t_j)$ and $R(t_j) \le R(t_j + \delta)$. So $R(t)$ has a local maximum at each jump point of $\psi(t)$ and $Q(t)$ changes sign from positive to negative. This can happen at most N times in $[-\pi.\pi)$ so $\psi(t)$ has at most N jump points and f lies in Π_n with $1 \le n \le N$. □

REMARK. Pinchuk's theorem may also be extended to cover the case of linear fuctionals involving evaluations at M points $\zeta_1, \cdots, \zeta_M$ in $\triangle$ by using

$$L(f) = \operatorname{Re} \sum_{j=0}^{N}\sum_{k=1}^{M} b_{jk}\frac{f^{(j+1)}(\zeta_k)}{(j+1)!}.$$

The function $Q(t)$ in the proof then has at most $2M(N+1)$ roots and L will attain its maximum only at points of Π_n for $1 \le n \le M(N+1)$, where this number may be reduced by one if $\zeta_k = 0$ for some k.

We now combine Proposition 1 and Proposition 2 to give the main result.

PROPOSITION 3. *There exist extreme points of $S_p(\alpha)$ which lie in Π_n for some $n \ge 2$.*

PROOF. By Proposition 2 the continuous linear functional L constructed in Proposition 1 attains its infimum $-X$ on the compact set $S_p(\alpha)$ only at points lying in Π_n for values $1 \leq n \leq N$. By Proposition 1 these cannot lie in Π_1. That is, $L^{-1}(\{-X\}) \cap S_p(\alpha) \subseteq \cup_{n=2}^{N} \Pi_n$.

The set $K = L^{-1}(\{-X\}) \cap \mathrm{co} S_p(\alpha)$ is a closed face of the compact convex set co $S_p(\alpha)$ so its extreme points are extreme points of co $S_p(\alpha)$. By Milman's Theorem they also lie in $S_p(\alpha)$ since this is compact. This shows that there exist extreme points of $S_p(\alpha)$ in Π_n where $2 \leq n \leq N$. □

References

1. Golusin, G. M., *On a variational method in the theory of analytic functions*, Leningrad Gos. Univ. Učen. Zap. 144. Ser. Mat. Nauk. **23** (1952), 85-101; Amer. Math. Soc. Transl. **18** (1961), 1-14.
2. Hallenbeck, D. J. and MacGregor, T. H., *Linear problems and convexity techniques in geometric function theory*, Pitman, London, 1984.
3. MacGregor, T. H., *Hull subordination and extremal problems for starlike and spirallike mappings*, Trans. Amer. Math. Soc. **183** (1973), 499-510.
4. Pearce, K, *A product theorem for $\mathcal{F}_p$ classes and an application*, Proc. Amer. Math. Soc. **84** (1982), 509-519.
5. Pinchuk, B., *On starlike and convex functions of order* α, Duke Math. J. **35** (1968), 721-734.
6. Slater, L. J., *Generalised Hypergeometric Functions*, Cambridge, 1966.

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