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SINGULARITY OF A COEFFICIENT MATRIX

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ABSTRACT. The interpolation of scattered data with radial basis functions is known for its good fitting. But if data get large, the coefficient matrix becomes almost singular. We introduce different knots and nodes to improve condition number of coefficient matrix The singulaity of new coefficient matrix is investigated here.

1. Introduction

Let $\{x_i : i = 1, 2, \ldots, n\}$ be a set of n distinct points in R^m. Let $\|\cdot\|$ be any norm on R^m, and consider the functions

$$g_j(x) = \|x - x_j\| \qquad j = 1, 2, \ldots, n.$$

If these "radial basis functions" are employed in interpolating arbitrary data at the points $x_1, x_2, \ldots x_n$, then the interpolating functions will be of the form $\sum_{j=1}^{n} c_j g_j(x)$ for unknown c_i's. Then we have to invert the coefficient matrix A whose elements are

$$A_{ij} = \|x_i - x_j\|.$$

It was proved by Schoenberg [3] that A is nonsingular if $\|\cdot\|$ is the Euclidean norm on R^m. See also [2] for more general results. In this paper we select a different set of points y_j to define the basis functions, but retain the points x_i as the interpolation nodes. The new coefficient matrix is given by

$$\bar{A}_{ij} = \|x_i - y_j\|.$$

For the simplicity we call y's knots and x's nodes. Although A is nonsingular for the Euclidean norm, $\bar{A}$ can be singular for certain configuration

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of knots. We investigate the position of knots in R^1 which ensures the nonsingularity of $\bar{A}$.

2. Main results

From now on we assume that $x_i, y_i \in R^1$ and $0 = x_1 < x_2 < \cdots < x_n = 1$ and $y_1 < y_2 < \cdots < y_n$. Also we assume that $y_i \in [x_1, x_n]$ for all i.

LEMMA 1. *If $y_m \in [x_j, x_{j+1}]$, then the m-th column of $\bar{A}$ is a linear and convex combination of the $j-th$ and the $(j+1)-st$ columns of A.*

PROOF. Let $h_j = x_j - x_{j-1}$ and $a_m = y_m - x_j$. Then the $m-th$ column of $\bar{A}$ has elements

$$|x_i - y_m| = |x_i - (\lambda x_j + \mu x_{j+1})| = |\lambda(x_i - x_j) + \mu(x_i - x_{j+1})|$$
$$= \begin{cases} \lambda(x_i - x_j) + \mu(x_i - x_{j+1}) & \text{if } i \geq j+1 \\ \lambda(x_j - x_i) + \mu(x_{j+1} - x_i) & \text{if } i \leq j \end{cases}$$

where

$$\lambda = \frac{x_{j+1} - y_m}{x_{j+1} - x_j} = 1 - \frac{a_m}{h_j}$$
$$\mu = \frac{y_m - x_j}{x_{j+1} - x_j} = \frac{a_m}{h_j}.$$

Hence the $m-th$ column of $\bar{A}$ has elements

$$\left(1 - \frac{a_m}{h_j}\right) A_{ij} + \left(\frac{a_m}{h_j}\right) A_{ij+1} \quad \square$$

Now we consider a configuration of the form $y_1 \in [x_1, x_2)$, $y_i \in (x_{i-1}, x_{i+1})$ for $i = 1, 2, \ldots, n$ and $y_n \in (x_{n-1}, x_n]$. Let C_i and $\bar{C}_i$ be the colums of A and $\bar{A}$ respectively. We introduce some notations by letting $e_i = |x_i - y_j|/h_i$ and $d_i = |x_i - y_i|/h_{i-1}$. Then we can easily see that

$$\bar{C}_i = d_i C_{i-1} + (1 - d_i) C_i \text{ if } y_i \in (x_{i-1}, x_i] \text{ and}$$
$$\bar{C}_i = (1 - e_i) C_i + e_i C_{i+1} \text{ if } y_i \in [x_i, x_{i+1})$$

Before we go further we need a definition and some remarks.

DEFINITION. Let (x_p, x_{p+1}) be an open subinterval containing no y's in it and (x_q, x_{q+1}) be the next such open subinterval. Then $[x_{p+1}, x_q]$ is called a *separated interval.*

Since the complement of the union of separated intervals contains no knots, each y_i belongs to some separated interval.

REMARK 1. The separated interval $[x_{p+1}, x_q]$ contains $y_{p+1}, \dots, y_q$.

REMARK 2. If $[x_{p+1}, x_q]$ is a separated interval, then exactly one of the intervals $[x_{p+1}, x_{p+2}), \cdots, [x_{q-2}, x_{q-1}), [x_{q-1}, x_q]$ contains two knots. Indeed, suppose there are more than one $[x_i, x_{i+1})$ (or$[x_{q-1}, x_q]$) containing two knots in $[x_{p+1}, x_q]$. Since every $[x_i, x_{i+1})$ (or x_{q-1}, x_q) contains at least one node, the separated interval $[x_{p+1}, x_q]$ contains more than $q - p$ knots.

LEMMA 2. *Let $[x_{p+1}, x_{p+m}]$ be any subinterval and $[x_{q+1}, x_{q+t}]$ be a separated interval with $p + m < q + 1$. If $\{\bar{C}_{p+1}, \dots, \bar{C}_{p+m}\}$ and $\{\bar{C}_{q+1}, \dots, \bar{C}_{q+t}\}$ are linearly independent sets respectively, then $\{\bar{C}_{p+1}, \dots, \bar{C}_{p+m}, \bar{C}_{q+1}, \dots, \bar{C}_{q+t}\}$ is a linearly independent set.*

PROOF. The columns $\bar{C}_{p+1}, \dots, C_{p+m}$ lie in $span\{C_p, C_{p+1}, \dots, C_{p+m}\}$ and $\bar{C}_{q+1}, \dots, \bar{C}_{q+t}$ in $span\{C_{q+1}, \dots, C_{q+t}\}$ since $y_{q+1} \in [x_{q+1}, x_{q+2})$. Since $p + m \neq q + 1$, the result follows immediately. □

THEOREM 1. *The matrix $\bar{A}$ is nonsingular if and only if $y_1 \in [x_1, x_2)$, $y_i \in (x_{i-1}, x_{i+1})$ for $i = 2, 3, \dots, n - 1$ and $y_n \in (x_{n-1}, x_n]$.*

PROOF. Suppose one of the knots does not satisfy the above condition. We will consider only $i = 2, 3, \dots, n-1$ since y_1 and y_n follow a similar analysis. Suppose $y_i \notin (x_{i-1}, x_{i+1})$. Then either $y_i \in [x_1, \dots, x_{i-1}]$ or $y_i \in [x_{i+1}, \dots, x_n]$. This implies that either $\bar{C}_1, \bar{C}_2, \dots, \bar{C}_i$ lie in $span\{C_1, C_2, \dots, C_{i-1}\}$ or $\bar{C}_i, \dots, \bar{C}_n$ in $span\{C_i + 1, \dots, C_n\}$ by Lemma 1. Both cases imply that the dimension of the column space of $\bar{A}$ is less than n.

Now assume that all knots satisfy the above condition. Let $[x_{p+1}, x_q]$ be any separated interval. By Lemma 2, we only need to show that $\{\bar{C}_{p+1}, \dots, \bar{C}_q\}$ is linearly independent. Let $[x_k, x_{k+1})$ ($[x_{q-1}, x_q]$ if $k =$

$q-1)$ be the subinterval containing two ys. Suppose that

$$\begin{aligned} a_{p+1}&\left[(1-e_{p+1})C_{p+1}+\epsilon_{p+1}C_{p+2}\right]+\\ &a_{p+2}\left[(1-\epsilon_{p+2})C_{p+2}+\epsilon_{p+2}C_{p+3}\right]+\dots\\ &+a_k\left[(1-e_k)C_k+\epsilon_kC_{k+1}\right]+a_{k+1}\left[d_{k+1}C_k+(1-d_{k+1})C_{k+1}\right]+\\ &a_{k+2}\left[d_{k+2}C_{K+1}+(1-D_{K+2})C_{K+1}\right]+\dots\\ &+a_q\left[d_qC_{q-1}+(1-d_q)C_q\right]=0. \end{aligned}$$

We will show that $a_i=0$ for all i.

After rearranging the above equation, we will get

$$\begin{aligned} &\left[a_{p+1}(1-e_{p+1})\right]C_{p+1}+\left[a_{p+1}\epsilon_{p+1}+a_{p+2}(1-\epsilon_{p+2})\right]C_{p+2}+\dots\\ &\quad+\left[a_{k-1}e_{k-1}+a_k(1-\epsilon_k)+a_{k+1}d_{k+1}\right]C_k\\ &+\left[a_ke_k+a_{k+1}(1-d_{k+1})+a_{k+2}d_{k+2}\right]C_{k+1}+\dots+a_k(1-d_k)C_k=0. \end{aligned}$$

(For $k=q-1, d_{k+2}=0$. For $k=p+1, e_{k-1}=0$.) Then $a_{p+1}=0$ since $1-\epsilon_{p+1}\neq 0$ and $a_q=0$ since $1-d_q\neq 0$. Successive steps show that $a_{p+2}=0, a_{q-1}=0,\dots,a_{k-1}=0, a_{k+2}=0$. We only need to show that $a_k=0$ and $a_{k+1}=0$. But

$$1-e_k-d_{k+1}=\frac{h_k-|x_k-y_k|-|x_{k+1}-y_{k+1}|}{h_k}\neq 0.$$

This implies that $a_k=0, a_{k+1}=0$. □

3. The inverse of an interpolating matrix

Using the inverse of A we can give an explicit form of $\bar{A}^{-1}$. The inverse of a matrix A is given in [1] by

$$\begin{pmatrix} \frac{h_1-1}{2h_1} & \frac{1}{2h_1} & 0 & 0 & 0 & \dots & \frac{1}{2}\\ \frac{1}{2h_1} & -\frac{h_1+h_2}{2h_1h_2} & \frac{1}{2h_2} & 0 & & \dots & 0\\ \vdots & & & \ddots & & & \vdots\\ 0 & & \dots & & \frac{1}{2h_{n-2}} & -\frac{h_{n-2}+h_{n-1}}{2h_{n-2}h_{n-1}} & \frac{1}{2h_{n-2}}\\ \frac{1}{2} & 0 & & \dots & 0 & \frac{1}{2h_{n-1}} & \frac{h_{n-1}-1}{2h_{n-1}} \end{pmatrix}$$

We let $R_i, \bar{R}_i$ be the $i-th$ rows of A^{-1} and $\bar{A}^{-1}$ respectively.

THEOREM 2. *Let $y_1 \in [x_1, x_2), y_i \in (x_{i-1}, x_{i+1})$ for $i = 2, \ldots, n-1$ and $y_n \in (x_n - 1, x_n]$. Let $[x_k, x_{k+1})$ be the unique subinterval containing two knots. If we assume that y_m belongs to the separated interval $[x_{p+1}, x_q]$, then*

$$\bar{R}_m = \begin{cases} \dfrac{1}{1-e_m} S_m & \text{if } p+1 \le m \le k-1 \\ \left(\dfrac{1-d_{k+1}}{1-e_k-d_{k+1}}\right) S_k - \left(\dfrac{d_{k+1}}{1-e_k-d_{k+1}}\right) T_{k+1} & \text{if } m = k \\ \left(\dfrac{1-e_k}{1-e_k-d_{k+1}}\right) T_{k+1} - \left(\dfrac{e_k}{1-e_k-d_{k+1}}\right) S_k & \text{if } m = k+1 \\ \dfrac{1}{1-d_m} T_m & \text{if } k+2 \le m \le q \end{cases}$$

where

$$S_m = \sum_{i=0}^{m-(p+1)} (-1)^i a_{m-i}^{m-1} R_{m-i}$$

$$T_m = \sum_{i=0}^{q-m} (-1)^i b_{m+1}^{m+i} R_{m+i}$$

and

$$a_s^t = \begin{cases} \dfrac{e_s \ldots e_t}{(1-e_s)\ldots(1-e_t)} \\ 1 \qquad \text{if } s > t \end{cases} \qquad b_s^t = \begin{cases} \dfrac{d_s \ldots d_t}{(1-d_s)\ldots(1-d_t)} \\ 1 \qquad \text{if } s > t \end{cases}$$

PROOF. We will show that $\bar{R}_m \cdot \bar{C}_i = \delta_{mi}$. If $i < p+1$ or $i > q$, $\bar{C}_i \in span\{C_1, \ldots, C_p\}$ or $\bar{C}_i \in span\{C_{q+1}, \ldots, C_n\}$ repectively. But $\bar{R}_m$ is a linear combination of $R_{p+1}, \ldots, R_q$. This implies that

$$\bar{R}_m \cdot \bar{C}_i = 0 \qquad \text{if } i < p+1 \qquad \text{or } i > q.$$

We assume that $p+1 \le i \le q$. We will treat the case $i = k$. Other cases ($p+1 \le i \le k-1, i = k+1, k+2 \le i \le q$) follow a similar analysis.

If $i = k$, then $\bar{C}_k = (1-e_k)C_k + e_k C_{k+1}$. If $p+1 \le m \le k-1$ or $k+2 \le m \le q$, $\bar{R}_m \cdot \bar{C}_k = 0$ since $\bar{R}_m$ is a linear combination of

$R_1, R_2 \ldots, R_{k-1}$ for $p+1 \leq m \leq k-1$ and a linear combination of $R_{k+2}, \ldots, R_q$ for $k+2 \leq m \leq q$.

If $m = k$,

$$\bar{R}_k \cdot \bar{C}_k = \left(\frac{1-d_{k+1}}{1-e_k-d_{k+1}}\right) \cdot (1-e_k) - \left(\frac{d_{k+1}}{1-e_k-d_{k+1}}\right) e_k$$
$$= \frac{1-e_k-d_{k+1}}{1-e_k-d_{k+1}} = 1$$

If $m = k+1$,

$$\bar{R}_{k+1} \cdot \bar{C}_k = \left(\frac{(1-e_k)e_k}{1-e_k-d_{k+1}}\right) - \left(\frac{e_k(1-e_k)}{1-e_k-d_{k+1}}\right) = 0. \quad \square$$

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