Comm. Korean Math. Soc. **10** (1995), No. 4, pp. 831–837

ON PERMUTATION GRAPHS OVER A GRAPH*

JAEUN LEE AND MOO YOUNG SOHN

ABSTRACT. In this paper, we introduce a permutation graph over a graph G as a generalization of both a graph bundle over G and a standard permutation graph, and study a characterization of a natural isomorphism and an automorphism of permutation graphs over a graph.

1. Introduction

A permutation graph was first introduced by Chartrand and Harary in [1] as a generalization of the Petersen graph. For completeness we recall the definition. Let G be a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For convenience, let $V(G) = \{u_1, u_2, \cdots, u_n\}$. For a permutation α in the symmetric group S_n on n elements, an *α-permutation graph* $P_\alpha(G)$ consists of two copies of G, say G_x and G_y, with vertex sets $V(G_x) = \{x_1, x_2, \cdots, x_n\}$ and $V(G_y) = \{y_1, y_2, \cdots, y_n\}$, along with edges $x_i y_{\alpha(i)}$ for $1 \le i \le n$.

Now, we introduce the notion of a permutation graph over a given graph G. Every edge of a graph G gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse edge to a directed edge $e = uv$. We denote the set of directed edges of G by $D(G)$. Following Gross and Tucker [4] a *(permutation) voltage assignment* ϕ of G is a function $\phi : D(G) \to S_n$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. Let $C^1(G; S_n)$ denote the set of all voltage assignments of G.

Let F be another graph with $V(F) = \{v_1, v_2, \cdots, v_{|V(F)|}\}$. For a voltage assignment $\phi \in C^1(G; S_{|V(F)|})$ of G, we construct a new graph

Received May 22, 1995. Revised August 1, 1995.

1991 AMS Subject Classification: 05C60, 05C25.

Key words and phrases: Permutation graphs, Natural isomorphisms and automorphisms.

*This research was supported by the Yeungnam University Research Grant in 1994.

$G \bowtie^{\phi} F$ as follows: $V(G \bowtie^{\phi} F) = V(G) \times V(F)$. Two vertices (u_i, v_h) and (u_j, v_k) are adjacent in $G \bowtie^{\phi} F$ if either $u_i u_j \in D(G)$ and $v_k = \phi(u_i u_j) v_h$ or $u_i = u_j$ and $v_h v_k \in E(F)$. This new graph $G \bowtie^{\phi} F$ is called an *F-permutation graph* over G. If G is the complete graph K_2 on two vertices, then the F-permutation graph $G \bowtie^{\phi} F$ is just a permutation graph. Note that the group of all graph automorphisms $\mathrm{Aut}\,(F)$ of F is a subgroup of $S_{|V(F)|}$. If ϕ takes its values in $\mathrm{Aut}\,(F)$, then the *F-permutation graph* $G \bowtie^{\phi} F$ is just an F-bundle $G \times^{\phi} F$ over G defined in [8], where the first coordinate projection $p^{\phi} : G \bowtie^{\phi} F \to G$ is the bundle projection.

In this paper, we study a characterization of a natural isomorphism and an automorphism of F-permutation graphs over a graph G.

2. Natural isomorphisms

Let Γ be a group of graph automorphisms of G. Given two voltage assignments ϕ and ψ in $C^1(G; S_{|V(F)|})$, two F-permutation graphs $G \bowtie^{\phi} F$ and $G \bowtie^{\psi} F$ over G are *naturally Γ-isomorphic* if there exist an isomorphism $\Phi : G \bowtie^{\phi} F \to G \bowtie^{\psi} F$ and an automorphism $\gamma \in \Gamma$ such that $p^{\psi} \circ \Phi = \gamma \circ p^{\phi}$, i.e., the following diagram

$$\begin{array}{ccc} G \bowtie^{\phi} F & \xrightarrow{\Phi} & G \bowtie^{\psi} F \\ {\scriptstyle p^{\phi}} \downarrow & & \downarrow {\scriptstyle p^{\psi}} \\ G & \xrightarrow[\gamma \in \Gamma]{} & G \end{array}$$

commutes. In this case, we call Φ a *natural Γ-isomorphism*. If Γ is the full group $\mathrm{Aut}\,(G)$, then we simply call $G \bowtie^{\phi} F$ and $G \bowtie^{\psi} F$ are *naturally isomorphic* and Φ a *natural isomorphism*.

THEOREM 1. *Let ϕ and ψ be two voltage assignments in $C^1(G; S_{|V(F)|})$, and Γ a group of graph automorphisms of G. Then two F-permutation graphs $G \bowtie^{\phi} F$ and $G \bowtie^{\psi} F$ are naturally Γ-isomorphic if and only if there exist an automorphism $\gamma \in \Gamma$ and a map $f : V(G) \to \mathrm{Aut}\,(F)$ such that $\psi(\gamma u_i \gamma u_j) = f(u_j) \phi(u_i u_j) f(u_i)^{-1}$ for all $u_i u_j \in D(G)$.*

PROOF. Let $G \bowtie^{\phi} F$ and $G \bowtie^{\psi} F$ be naturally Γ-isomorphic with an isomorphism $\Phi : G \bowtie^{\phi} F \rightarrow G \bowtie^{\psi} F$. Then $\Phi|_{(p^{\phi})^{-1}(u)} : (p^{\phi})^{-1}(u) \rightarrow (p^{\psi})^{-1}(\gamma(u))$ is an isomorphism for all $u \in V(G)$ and for some $\gamma \in \Gamma$. Now, we define $f : V(G) \rightarrow \text{Aut}(F)$ by $f(u) = \Phi|_{(p^{\phi})^{-1}(u)}$ for all $u \in V(G)$. If (u_i, v_h) is joined to (u_j, v_k) in $G \bowtie^{\phi} F$, then $\phi(u_iu_j)(v_h) = v_k$ and $(\gamma u_i, f(u_i)(v_h))$ is joined to $(\gamma u_j, f(u_j)(v_k))$ in $G \bowtie^{\psi} F$. Thus $\psi(\gamma u_i \gamma u_j) = f(u_j)\phi(u_iu_j)f(u_i)^{-1}$ for all $u_iu_j \in D(G)$. Conversely, let's define $\Phi : G \bowtie^{\phi} F \rightarrow G \bowtie^{\psi} F$ by $\Phi(u_i, v_h) = (\gamma u_i, f(u_i)(v_h))$ for any (u_i, v_h) in $V(G \bowtie^{\phi} F)$. If (u_i, v_h) is joined to (u_j, v_k) in $G \bowtie^{\phi} F$, then $\phi(u_iu_j)(v_h) = v_k$ and $\Phi(u_i, v_h) = (\gamma u_i, f(u_i)(v_h))$ is joined to $\Phi(u_j, v_k) = (\gamma u_j, f(u_j)(v_k))$. Thus Φ is the desired isomorphism to complete the proof.

THEOREM 2. *Let ϕ and ψ be two voltage assignments in $C^1(G; S_{|V(F)|})$, and Γ a group of automorphisms of G. Let η be an automorphism of F and $u \in V(G)$. Then there exists a natural Γ-isomorphism $\Phi : G \bowtie^{\phi} F \rightarrow G \bowtie^{\psi} F$ with $\Phi|_{(p^{\phi})^{-1}(u)} = \eta$ if and only if there exists an automorphism $\gamma \in \Gamma$ such that $\psi(\gamma W)\eta\phi(W)^{-1} = \eta$ for any closed walk W based at u, and $\psi(\gamma P)\eta\phi(P)^{-1} \in \text{Aut}(F)$ for any path P beginning at u.*

PROOF. The necessity comes from Theorem 1. Now, we aim to prove the sufficiency. To get a natural Γ-isomorphism, we first define $f : V(G) \rightarrow \text{Aut}(F)$ as follows: Let $f(u) = \eta$. Since G is connected, for each $u_i \in V(G)$ there exists a path P from u to u_i. Let $f(u_i) = \psi(\gamma P)\eta\phi(P)^{-1}$. Then the value $f(u_i)$ depends only on the vertex u_i, because $\phi(\gamma W)\eta\psi(W)^{-1} = \eta$ for any closed walk based at u. Then for each $u_iu_j \in D(G)$

$$\begin{aligned} f(u_j)\phi(u_iu_j)f(u_i)^{-1} &= \psi(\gamma P_j)\,\eta\,\phi(P_j)^{-1}\,\phi(u_iu_j)\phi(P_i)\,\eta^{-1}\,\psi(\gamma P_i)^{-1} \\ &= \psi(\gamma P_j)\,\psi(\gamma P_i)^{-1} \\ &= \psi(\gamma u_i \gamma u_j). \end{aligned}$$

where P_i is a path from u to u_i and P_j is the path obtained by adding the edge u_iu_j to the path P_i. Now, theorem comes from Theorem 1.

For a voltage assignment ϕ in $C^1(G; S_{|V(F)|})$ and a walk $w = e_1 \cdots e_m$ in G, we define the *net ϕ-voltage* $\phi(W)$ of W by the value $\phi(e_m)\phi(e_{m-1})$

$\cdots\phi(e_1)$. Then the set $\mathcal{L}_u(\phi)$ of all net ϕ-voltages of the closed walks based at $u \in V(G)$ is a subgroup of $S_{|V(F)|}$. We call $\mathcal{L}_u(\phi)$ the ***local voltage group*** of ϕ at u.

COROLLARY 1. *Let ϕ be a voltage assignment in $C^1(G; S_{|V(F)|})$ and Γ a group of automorphisms of G. Then $G \bowtie^{\phi} F$ and $G \times F$ are naturally Γ-isomorphic if and only if the local voltage group $\mathcal{L}_u(\phi)$ of ϕ is trivial for each u in $V(G)$ and $\phi(u_iu_j) \in \mathrm{Aut}\,(F)$ for each $u_iu_j \in D(G)$.*

PROOF. Let ψ be the trivial voltage assignment of G, i.e., for each $e \in D(G)$, $\psi(e)$ is the identity of $S_{|V(F)|}$. Then $G \bowtie^{\psi} F$ is naturally $\{1\}$-isomorphic to the product $G \times F$ of G and F. Now, corollary comes from Theorem 2.

Notice that if G is a tree, then the local voltage group of any voltage assignment ϕ in $C^1(G; S_{|V(F)|})$ is trivial. Now, the following also comes from Theorem 2.

COROLLARY 2. *Let Γ be a group of automorphisms of a tree G, and let ϕ and ψ be two voltage assignments in $C^1(G; S_{|V(F)|})$. Let η be an automorphism of F and $u \in V(G)$. Then there exists a natural Γ-isomorphism $\Phi : G \bowtie^{\phi} F \to G \bowtie^{\psi} F$ with $\Phi|_{(p^{\phi})^{-1}(u)} = \eta$ if and only if there exists $\gamma \in \Gamma$ such that $\psi(\gamma P)\eta\phi(P)^{-1} \in \mathrm{Aut}\,(F)$ for any path P beginning at u.*

It is clear that if F is disconnected, then there exist two voltage assignments ϕ and ψ of G such that $G \bowtie^{\phi} F$ and $G \bowtie^{\psi} F$ are isomorphic by a nonnatural isomorphism.

3. Natural automorphisms

In this section, we study the natural automorphism of an F-permutation graph. Let $\mathrm{Aut}\,^{N}_{\Gamma}(G \bowtie^{\phi} F)$ denote the group of all natural Γ-automorphisms Φ of $G \bowtie^{\phi} F$, that is, there exists an automorphism $\gamma \in \Gamma$ such that $p^{\phi} \circ \Phi = \gamma \circ p^{\phi}$.

The following comes from Theorem 2.

COROLLARY 3. *Let ϕ be a voltage assignment in $C^1(G; S_{|V(F)|})$ and η an automorphism of F. Let Γ be a group of automorphisms of G and $u \in V(G)$. Then there exists a natural Γ-automorphism $\Phi : G \bowtie^{\phi} F \to G \bowtie^{\phi} F$ with $\Phi|_{(p^{\phi})^{-1}(u)} = \eta$ if and only if there exists an automorphism $\gamma \in \Gamma$ such that $\eta = \phi(\gamma W)^{-1}\, \eta\, \phi(W)$ for any closed walk W based at u, and $\eta \in \phi(\gamma P)^{-1}\mathrm{Aut}\ (F)\phi(P)$ for any path P beginning at u.*

Let $\mathcal{A}$ be a group. For a subset S of $\mathcal{A}$, let $\mathbf{Z}(S)$ denote the centralizer of S.

COROLLARY 4. *Let ϕ be a voltage assignment in $C^1(G; S_{|V(F)|})$ and η an automorphism of F. Let $u \in V(G)$. Then there exists a natural $\{1\}$-automorphism $\Phi : G \bowtie^{\phi} F \to G \bowtie^{\phi} F$ with $\Phi|_{(p^{\phi})^{-1}(u)} = \eta$ if and only if $\eta \in \mathbf{Z}(\mathcal{L}_u(\phi)) \cap \phi(P)^{-1}\mathrm{Aut}\ (F)\phi(P)$ for any path P beginning at u. Moreover, if G is a tree, then there exists a natural $\{1\}$-automorphism $\Phi : G \bowtie^{\phi} F \to G \bowtie^{\phi} F$ with $\Phi|_{(p^{\phi})^{-1}(u)} = \eta$ if and only if $\eta \in \phi(P)^{-1}\mathrm{Aut}\ (F)\phi(P)$ for any path P beginning at u.*

From Corollary 4, we can deduce that $\mathrm{Aut}\,^{N}_{\{1\}}(G \bowtie^{\phi} F)$ is isomorphic to

$$\mathrm{Aut}\,(F) \bigcap \mathbf{Z}(\mathcal{L}_u(\phi)) \bigcap \left(\bigcap_{P} \phi(P)^{-1}\mathrm{Aut}\ (F)\, \phi(P) \right),$$

where u is a fixed vertex of G and P runs over all paths beginning at u.

Let Φ be a natural automorphism in $\mathrm{Aut}\,^{N}_{\Gamma}(G \bowtie^{\phi} F)$. Then, by Theorem 1, there exists a unique pair $(\gamma_{\Phi}, f_{\Phi})$ with $\gamma_{\Phi} \in \Gamma$ and $f_{\Phi} : V(G) \to \mathrm{Aut}\,(F)$ such that $\Phi(u, v) = (\gamma u, f(u)(v))$ for each $(u, v) \in V(G \bowtie^{\phi} F)$. We call such a pair $(\gamma_{\Phi}, f_{\Phi})$ the *canonical factorization* of Φ.

For convenience, let $C^0(G; \mathrm{Aut}\,(F))$ be the set of all maps from $V(G)$ to $\mathrm{Aut}\,(F)$. Then $C^0(G; \mathrm{Aut}\,(F))$ is a group under pointwise multiplication, that is, for each u in $V(G)$, $(f_1 f_2)(u) = f_1(u) f_2(u)$. Let Γ be a group of automorphisms of G. Define a Γ action on $C^0(G; \mathrm{Aut}\,(F))$ by $(\gamma f_1)(u) = f_1(\gamma^{-1} u)$ for each $u \in V(G)$. This action determines a group structure on the set $\Gamma \times C^0(G; \mathrm{Aut}\,(F))$, that is,

$$(\gamma_1, f_1)(\gamma_2, f_2) = (\gamma_1 \gamma_2, (\gamma_2^{-1} f_1) f_2)\,.$$

Let $\Gamma * \mathrm{Aut}\,(F)$ denote this group.

Define $\theta : \mathrm{Aut}_{\Gamma}^{N}(G \bowtie^{\phi} F) \to \Gamma * \mathrm{Aut}\,(F)$ by $\theta(\Phi) = (\gamma_{\Phi}, f_{\Phi})$. Then θ is a group homomorphism. In fact, Since $\Psi(u,v) = (\gamma_{\Psi} u, f_{\Psi}(u)(v))$ and $\Phi(u,v) = (\gamma_{\Phi} u, f_{\Phi}(u)(v))$,

$$\begin{aligned}
(\Phi \circ \Psi)(u,v) &= \Phi\left(\gamma_{\Psi} u, f_{\Psi}(u)(v)\right) \\
&= \left(\gamma_{\Phi}\gamma_{\Psi} u, f_{\Phi}(\gamma_{\Psi} u)(f_{\Psi}(u)(v))\right) \\
&= \left(\gamma_{\Phi}\gamma_{\Psi} u, (\gamma_{\Psi}^{-1} f_{\Phi})(u)(f_{\Psi}(u)(v))\right) \\
&= \left(\gamma_{\Phi}\gamma_{\Psi} u, \left((\gamma_{\Psi}^{-1} f_{\Phi}) f_{\Psi}\right)(u)(v)\right),
\end{aligned}$$

for each (u,v) in $V(G \bowtie^{\phi} F)$. It implies that $\gamma_{\Phi \circ \Psi} = \gamma_{\Phi}\gamma_{\Psi}$ and $f_{\Phi \circ \Psi} = \left(\gamma_{\Psi}^{-1} f_{\Phi}\right) f_{\Psi}$. Thus we have

$$\theta(\Phi \circ \Psi) = \left(\gamma_{\Phi}\gamma_{\Psi}, \left(\gamma_{\Psi}^{-1} f_{\Phi}\right) f_{\Psi}\right) = (\gamma_{\Phi}, f_{\Phi})(\gamma_{\Psi}, f_{\Psi}) = \theta(\Phi)\theta(\Psi).$$

Now, the following comes from the uniqueness of the pair $(\gamma_{\Phi}, f_{\Phi})$ for a natural automorphism Φ.

THEOREM 3. *Let ϕ be a voltage assignment in $C^1(G; S_{|V(F)|})$ and Γ a group of graph automorphisms of G. Then $\mathrm{Aut}_{\Gamma}^{N}(G \bowtie^{\phi} F)$ is isomorphic to a subgroup of $\Gamma * \mathrm{Aut}\,(F)$.*

Let Γ be a group of automorphisms of G. A natural automorphism Φ of an F-permutation graph $G \bowtie^{\phi} F$ is an element of $\mathrm{Aut}_{\Gamma}^{N}(G \bowtie^{\phi} F)$ if and only if γ_{Φ} is an element of Γ. Since $\gamma_{\Phi \circ \Psi \circ \Phi^{-1}} = \gamma_{\Phi}\gamma_{\Psi}\gamma_{\Phi}{}^{-1}$, we have the following Theorem.

THEOREM 4. *Let Γ be a normal subgroup of $\mathrm{Aut}\,(G)$ and ϕ a voltage assignment in $C^1(G; S_{|V(F)|})$. Then $\mathrm{Aut}_{\Gamma}^{N}(G \bowtie^{\phi} F)$ is a normal subgroup of $\mathrm{Aut}_{\mathrm{Aut}\,(G)}^{N}(G \bowtie^{\phi} F)$. In particular, $\mathrm{Aut}_{\{1\}}^{N}(G \bowtie^{\phi} F)$ is a normal subgroup of $\mathrm{Aut}_{\mathrm{Aut}\,(G)}^{N}(G \bowtie^{\phi} F)$.*

ACKNOWLEDGMENT. The authors would like to express their gratitude to the referee for valuable comments.

References

1. G. Chartrand and F. Harary, *Planar permutation graphs*, Ann. Inst. Henri Poincaré **III** (1967), 433-438.

2. W. Dörfler, *On mapping graphs and permutation graphs*, Math. Slovaca **28** (1978), 277-288.
3. ______, *Automorphisms and isomorphisms of permutation graphs*, Colloques Inter., C.N.R.S., No. 260, Problèmes Combinatoires et Théorie des Graphes (1978), 109-110.
4. J. L. Gross and T. W. Tucker, *Generating all graph coverings by permutation voltage assignments*, Discrete Math. **18** (1977), 273-283.
5. J. H. Kwak and J. Lee, *Isomorphism classes of graph bundles*, Canad. J. Math. **XLII** (1990), 747-761.
6. ______, *Counting some finite-fold coverings of a graph*, Graphs and Combinatorics **8** (1992), 277-285.
7. ______, *Isomorphism classes of cycle permutation graphs*, Discrete Math. **105** (1992), 131-142.
8. B. Mohar, T. Pisanski and M. Škoviera,, *The maximum genus of graph bundles*, Europ. J. Combinatorics **9** (1988), 215-224.
9. R. Ringeisen, *On cycle permutation graphs*, Discrete Math. **51** (1984), 265-275.
10. S. Stueckle, *On natural isomorphisms of cycle permutation graphs*, Graphs and Combinatorics **4** (1988), 75-85.

Jaeun Lee
Department of Mathematics
Yeungnam University
Kyongsan, 712-749 Korea

Moo Young Sohn
Department of Mathematics
Changwon National University
Changwon, 641-773 Korea