Comm. Korean Math. Soc. 10 (1995), No. 4, pp. 811–813

GENERALIZATIONS OF THE QUASI-INJECTIVE MODULE

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ABSTRACT. The purpose of this paper is to prove the divisibility of a direct injective module and every closed submodule of a π-injective module M is a direct summand of M.

1. Introduction

Let R be a ring with unity and let M be a unitary R-module. A module M is said to be quasi-injective (pseudo-injective) if for every submodule N of M, every homomorphism (monomorphism) of N into M can be extended to an endomorphism of M. A module M is direct injective if, given any direct summand A of M, an injection $i : A \longrightarrow M$ and every monomorphism $f : A \longrightarrow M$, there exists an endomorphism g of M such that $g \circ f = i$. A module M is said to be π-injective if it satisfies (i) for every submodule A of M, there exists a submodule B of M such that B is a direct summand of M and A is an essential submodule of B (ii) for every direct summand A, B of M, $A \cap B = O$ implies that $A \oplus B$ is a direct summand of M. We know that the pseudo-injective, direct injective and π-injective modules are generalizations of a quasi-injective module from the above definitions. We denote the singular submodule cl(0) of M by $Z(M)$ and clcl(0) by $Z_2(M)$. If a maximal submodule M' of M has the property $M' \cap N = O$ for a submodule N of M, we call M' a complement of N in M. Harada proved that every closed submodule of a quasi-injective module M is a direct summand of M.[1]

Received April 1, 1995. Revised September 28, 1995
1991 AMS Subject Classification: 16D50.
Key words and phrases: Direct injective module, π-injective module.
This paper was supported by research fund of Dong-A University, 1995.

In this paper, we show that the divisibility of an injective module is generalized to the divisibility of a direct injective module, we extend the above result of Harada to the case π-injective module M.

2. Results

THEOREM 2.1. *Every direct injective module M is divisible.*

PROOF. Let $M = A \oplus B$. Then we can define $f_r : A \longrightarrow M$ by $f_r(a) = ra$ for a fixed nonzero divisor $r \in R$ and all $a \in A$. First, we will show that f_r is well-defined. If $a = b$ for every $a, b \in A$, then $ra = rb$ and $f_r(a) = f_r(b)$. The next thing to verify is that f_r is monomorphism. Let $f_r(a) = f_r(b)$ for every $a, b \in A$, then $ra = rb$, $ra - rb = 0$ and $r(a-b) = 0$. Since r is a nonzero divisor, $a = b$. Since M is direct injective, for the inclusion $i : A \longrightarrow M$ and every monomorphism $f_r : A \longrightarrow M$ for all nonzero divisor $r \in R$, there exists an endomorphism g of M such that $g \circ f_r(a) = i(a) = a = g(ra) = rg(a)$. By using the directivity of M we have $g(a) \in A$. That is, there exists $g(a) \in A$ such that $rg(a) = a$ for all $a \in A$. Thus A is divisible. Since the sum of divisible modules is divisible, M is divisible.

THEOREM 2.2. *Pseudo-injective module is direct injective.*

PROOF. Let $M = A \oplus B$. Then from the definition of pseudo-injective module, an injection $i : A \longrightarrow M$ can be extended to an endomorphism g of M.

THEOREM 2.3. *Let M be π-injective module. Then every closed submodule N is a direct summand of M.*

PROOF. Let N be a closed submodule and C a complement of N in M. Put $M' = N \oplus C$. Since M is π-injective, by using [2, Proposition 1.1], for a projection $p : M' \longrightarrow N$, there exists an endomorphism g of M such that $g|M' = p$. Since $C \subset \operatorname{Ker} g$, $\operatorname{Ker} g \cap N = O$ and C is a complement of N, $\operatorname{Ker} g = C$. By [1, Lemma 1.4] $\operatorname{cl}(M') = M$ and hence there exists an essential left ideal I for any $m \in M$ such that $Im \subset M'$. Thus $Ig(m) = g(Im) \subset N$. Since $\operatorname{cl} N = N, g(m) \in N$. This implies $g(M) = N$. Hence $M = \operatorname{Ker} g \oplus g(M) = C \oplus N$.

COROLLARY 2.4. *A closed submodule of a π-injective module M is π-injective.*

PROOF. Suppose that N is closed submodule of a π-injective module M. Then by Theorem 2.3 N is a direct summand of M. We will show that N is π-injective. Since M is π-injective, for every submodule $A \oplus B$ of $N(\subset M)$ and a canonical projection $p : A \oplus B \longrightarrow A$, there exists an endomorphism of M such that $g|_{A \oplus B} = p$. Moreover, we obtain $g(M) = N$ from the result of Theorem 2.3. This means that g is an endomorphism of N such that $g|_N|_{A \oplus B} = p$. Hence by [2, Proposition 1.1] N is π-injective.

COROLLARY 2.5. *If S is any submodule of M, then there exists a π-injective essential extension of S contained in M.*

PROOF. By Zorn's Lemma S is contained in a closed submodule N which is an essential extension of S and by Corollary 2.4, N is π-injective.

COROLLARY 2.6. *Each minimal π-injective extension of a module M is an essential extension of M.*

PROOF. This is an immediate consequence of Corollary 2.5.

THEOREM 2.7. *Let M be a π-injective module and M' a maximal submodule with $Z(M') = O$. Then $M = M' \oplus Z_2(M)$.*

PROOF. According to [1, Lemma 1.6] $Z(M') = M' \cap Z(M) = O$, $Z(M') = \mathrm{cl}(0) = O = \mathrm{clcl}(0) = Z_2(M')$ and $Z_2(M') = M' \cap Z_2(M) = O$. Hence we obtain that $M' \cap Z_2(M) = O$ and M' is a complement of $Z_2(M)$. Hence $M = M' \oplus Z_2(M)$.

References

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