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ON CANCELLATION IDEALS

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ABSTRACT. In this paper we characterize cancellation ideals in terms of multiplication ideals. Especially, we find a condition for an ideal generated by three elements to be a cancellation ideal.

1. Introduction and Preliminary

Let R be a commutative ring with an identity. An ideal A of R is called a *multiplication ideal* if for every ideal $B \subseteq A$ there exists an ideal C of R such that $B = AC$. Recall that an ideal A in the ring R is said to be a *cancellation ideal* if, for all ideals B, C of R, $AB \subseteq AC$ implies that $B \subseteq C$. The ideal A is said to be a *weak cancellation ideal* if, for all ideals B, C of R, $AB \subseteq AC$ implies that $B \subseteq C + ann(A)$. Thus every principal ideal is a weak cancellation ideal. It is well known and easy to prove that every invertible ideal is a cancellation ideal. If A and B are any ideals in R, We put

$$[A : B] = \{x \in R | xB \subseteq A\}.$$

If $A = (a)$ or $B = (b)$ is a principal ideal;then we put $[a : B]$ for $[(a) : B]$ and $[A : b]$ for [A:(b)] and [a:b] for $[(a):(b)]$. In this paper we characterize cancellation ideals in terms of multiplication ideals. Especially, we find a condition for an ideal generated by three elements to be a cancellation ideal.

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2. Cancellation ideals

The following result shows that finitely generated faithful multiplication ideals in any ring R are cancellation ideals

THEOREM 1. *Let A be a finitely generated faithful multiplication ideal in the ring R. Then A is a cancellation ideal*

PROOF. Suppose A is generated by $(a_1, ..., a_n)$ and $AB \subseteq AC$ for any ideals B, C of R. Since A is a multiplication ideal, there exists an $n \times n$ matrix $M = (r_{ij})$ with entries in R such that

(i) $U = UM$
(ii) $r_{ij}a_k = r_{ik}a_j$
(iii) Trace$M = 1$ Where U is the n-vector $(a_1, \cdots, a_n)$ in R^n, and R^n is the direct product of n copies of R [see[6], Theorem 1.1]

Let $b \in B$. Then, for each j, $1 \leq j \leq n$, $ba_j \in AC$, hence

$$ba_j = \sum_{j=1}^{n} c_{ij}a_i$$

Where $c_{ij} \in C$. This implies

$$\sum_{i \neq j} c_{ij}a_i + (c_{jj} - b)a_j = 0$$

Thus the n-vector $(c_{1j}, c_{2j}, \cdots, c_{jj} - b, \cdots, c_{nj}) \in U$. By the properties of the matrix M [6], it follows that

$$\sum_{i \neq j} c_{ij}r_{ji} + (c_{jj} - b)r_{jj} \in ann(A)$$

Since A is faithful,

$$\sum_{i \neq j} c_{ij}r_{ji} + (c_{jj} - b)r_{jj} = 0$$

and hence $br_{jj} = \sum_{i=1}^{n} c_{ij}r_{ji} \in C$. Since Trace$M = 1$,

$$\sum_{j=1}^{n} r_{jj} = 1$$

and hence $b \in C$. Therefore A is a cancellation ideal.

REMARK 2. Theorem 1 shows that if A is non-faithful, A is a weak cancellation ideal and hence any finitely generated multiplication ideals in any ring R are weak cancellation ideals. Theorem 1 has several other consequences which we wish to mention.

COROLLARY 3. *If* (a,b), (a,c), *and* (b,c) *are multiplication ideals in* R *and the annihilator of* (a,b,c) *is trivial, then* (a,b,c) *is a cancellation ideal.*

PROOF. By [[6], Theorem 1.1(3)], $[a:b]+[b:a]=R$, $[a:c]+[c:a]=R$ and $[b:c]+[c:b]=R$. Then $[a:b]+[c:b]+[a:c]+[b:c]+[b:a]+[c:a]=R$. Since $[a:b]+[c:b]\subseteq[(a,c):b]$, we get $[(a,b):c]+[(a,c):b]+[(b,c):a]=R$. Now, for each maximal ideal M, $[(a,b)R_M:cR_M]+[(a,c)R_M:bR_M]+[(b,c)R_M:aR_M]=R_M$. So $1=x_1+x_2+x_3$ where $x_1\in[(a,b)R_M:cR_M]$, $x_2\in[(a,c)R_M:bR_M]$, $x_3\in[(b,c)R_M:aR_M]$. Since R_M is a local ring, it follows that one of the $x_i's$ is unit, say x_1. Then $[(a,b)R_M:cR_M]=R_M$. This implies $cR_M\subseteq(a,b)R_M$. Since (a,b) is a multiplication ideal, it is locally principal i.e $(a,b)R_M$ is principal (see [1]). It follows that $(a,b,c)R_M$ is a principal ideal. This shows that (a,b,c) is locally principal and hence (a,b,c) is a multiplication ideal [see[6], Theorm 1.1] By Theorem 1, (a,b,c) is a cancellation ideal.

COROLLARY 4. *Let* A *be a finitely generated projective ideal in* R *with* $ann(A)=(1-e)$, *where* $e=e^2$. *If* $AB\subseteq AC$ *for some ideals* B, C *in* R, *then* $eB=eC$.

PROOF. Since every projective ideal is a multiplication ideal [see[7], Theorem 1], $B\subseteq C+(1-e)$ by Theorem 1 and so $eB\subseteq eC$.

COROLLARY 5. *Let* A *be an ideal of* R *as in Theorem 1. Then* $[AB:A]=B$ *for each ideal* B *of* R.

PROOF. Let $x\in[AB:A]$ and consider an ideal (x) of R. Then $(x)A=A(x)\subseteq AB$ for each ideal B of R. By Theorem 1, $(x)\subseteq B$. Thus $[AB:A]\subseteq B$. The other inclusion is obvious.

The following result is taken from $[4, p.43ex.23]$. This exercise is concerned with our interest.

THEOREM 6. *Let R be a domain in which every ideal generated by two elements is invertible. Then every nonzero finitely generated ideal is a cancellation ideal.*

PROOF. Let P be any prime ideal of R. We show that R_p is a valuation domain by showing that each pair of principal ideals of R_p compares under $\subseteq$. Let a and b be two nonzero elements of R_p. We can write $a = a_1/s_1$, $b = b_1/s_2$ where $a_1, b_1 \in R$ and $s_1 . s_2 \in R - P$. Then $aR_p = a_1R_p$ and $bR_p = b_1R_p$. By hypothesis, $(a_1 . b_1)$ is invertible in R and so $(a_1, b_1)R_p$ is invertible [see[4], Theorem 61]. This shows that $(a_1, b_1)R_p = a_1R_p$ or $(a_1, b_1)R_p = b_1R_p$ [see[3], Proposition7.4] If say, $(a_1, b_1)R_p = a_1R_p$, then $bR_p \subseteq aR_p$ so that R_p is a valuation domain. By [4, *Theorem*64], R is Prüfer.i.e every nonzero finitely generated ideal is invertible and hence a cancellation ideal.

REMARK 7. It is well known and easy to prove that an ideal A of R is invertible if and only if it is a multiplication ideal which contains a regular element of R and so the theorem remains true if "invertible" is replaced by "multiplication ideal".

References

1. D. D. Anderson, *Multiplication ideals, multiplication rings and the ring $R(x)$*, Canad. J. Math. **28** (1976), 760-768.
2. Z. A. El-Bast and P. F. Smith, *Multiplication modules*. Comm. Algebra. **16** (1988), 755-779.
3. R, Gilmer, *Multiplicative ideal theory*, New York, 1972.
4. I. Kaplansky, *Commutative rings*, Allyn and Bacon, 1970.
5. A. G. Naoum, $\frac{1}{2}$ *Cancellation modules*, Kyungpook Math J (to appear).
6. A. G. Naoum and M. A. K. Hansan, *The residual of finitely generated multiplication modules*, Arch. Math **46** (1986), 225-230.
7. W. W. Smith, *Projective ideals of finite type*, Canad. J. Math. **21** (1969), 1057-1061.

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