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THE EXISTENCE RESULT OF CERTAIN SINGULAR BOUNDARY VALUE PROBLEM

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ABSTRACT. We prove the existence of solution for a Dirichlet singular boundary value problem. The proofs are based on the method of upper and lower solutions.

1. Introduction

In this paper, we are concerned with a singular boundary value problem of the form:

$$(1_s)\qquad \begin{gathered} u''(t) + st^{-\frac{3}{2}}e^{u(t)} = 0, \\ u(0) = 0 = u(1), \end{gathered}$$

where s is a positive real parameter.

This problem is an example arising from the study of the structure of diffusion flame near ignition. Choi [1] has shown that there exists a positive real number s_o such that (1_s) has no solution for $s > s_o$ or at least one solution for $s < s_o$. He employed a shooting method to prove this and gave some numerical results. According to the numerical results, one may expect that there exists $s_o > 0$ such that (1_s) has no solution for $s > s_o$, at least one solution for $s = s_o$, and at least two solutions for $s < s_o$. But he has not been successful in showing this.

For the existence of multiple solutions, it is a routine procedure to divide the set containing possible solutions into two or more sectors and show the existence on each sector. Since a sector can be obtained by

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means of upper and lower solutions, it is interesting to consider upper and lower solutions of problem (1_s) for the multiplicity.

The purpose of this paper is to prove Choi's result using the method of upper and lower solutions. It gives quite different proof from Choi and furthermore extends his result as follows;
There exists $s_o > 0$ such that (1_s) has no solution for $s > s_o$ and at least one solution for $s \leq s_o$.

2. Methods of upper and lower solutions.

We present a theorem on upper and lower solutions for the singular problem we are dealing with. Consider the problem

$$(2) \qquad \begin{aligned} &u''(t) + f(t, u(t)) = 0, \\ &u(0) = a, \ u(1) = b, \end{aligned}$$

where $f : D \to \mathbf{R}$ is a continuous function and $D \subset (0,1] \times \mathbf{R}$. A solution $u(\cdot)$ of (2) means a function $u \in C([0,1], \mathbf{R}) \cap C^2((0,1], \mathbf{R})$ such that $(t, u(t)) \in D$ for all $t \in (0,1]$ and $u''(t) + f(t, u(t)) = 0$ for all $t \in (0,1]$ with $u(0) = a$ and $u(1) = b$.

DEFINITION 1. $\alpha \in C([0,1], \mathbf{R}) \cap C^2((0,1], \mathbf{R})$ is called a *lower solution* of (2) if $(t, \alpha(t)) \in D$ for all $t \in (0,1]$ and

$$\begin{aligned} &\alpha''(t) + f(t, \alpha(t)) \geq 0, \ t \in (0,1] \\ &\alpha(0) \leq a, \ \alpha(1) \leq b. \end{aligned}$$

$\beta \in C([0,1], \mathbf{R}) \cap C^2((0,1], \mathbf{R})$ is called an *upper solution* of (2) if $(t, \beta(t)) \in D$ for all $t \in (0,1]$ and

$$\begin{aligned} &\beta''(t) + f(t, \beta(t)) \leq 0, \ t \in (0,1] \\ &\beta(0) \geq a, \ \beta(1) \geq b. \end{aligned}$$

We define the set $D_\alpha^\beta = \{(t, x) \in (0,1] \times \mathbf{R} : \alpha(t) \leq x \leq \beta(t)\}$. The following theorem is a slight modification of Theorem 1 in Habets and Zanolin [2]. This modification is straightforward to problem (1_s) and the proof can be done by obvious changes from [2].

THEOREM 1. *Let α, β be a lower and an upper solution for (2) such that*
(a_1) $\alpha(t) \leq \beta(t)$ *for all* $t \in [0,1]$ *and suppose that*
(a_2) $D_\alpha^\beta \subset D$,
assume also that there is a function $h \in C((0,1], \mathbf{R}^+)$ *such that*
(a_3) $|f(t,x)| \leq h(t)$ *for all* $(t,x) \in D_\alpha^\beta$ *and*
(a_4) $\int_0^1 sh(s)ds < \infty$.
Then (2) has at least one solution $\tilde{u}(\cdot)$ *such that*

$$\alpha(t) \leq \tilde{u}(t) \leq \beta(t) \qquad \text{for all} \quad t \in [0,1].$$

3. Main result

Let us consider the following nonlinear Dirichlet singular boundary value problem

$$(1_s) \qquad \begin{aligned} u''(t) + st^{-\frac{3}{2}} e^{u(t)} &= 0, \\ u(0) = 0 &= u(1), \end{aligned}$$

where $s > 0$ is a real parameter. Our aim is to prove that there exists $s_o > 0$ such that (1_s) has no solution or at least one solution according to $s > s_o$ or $s \leq s_o$ respectively. We first notice that every solution $u(t)$ of (1_s) is convex upward on $[0,1]$ and $u(t) > 0$ on $(0,1)$, since $u''(t) = -st^{-\frac{3}{2}} e^{u(t)} < 0$ on $(0,1)$ and $u(0) = 0 = u(1)$. We need some lemmas for the result.

LEMMA 1. *Consider*

$$(3_k) \qquad \begin{aligned} u''(t) + ke^{u(t)} &= 0, \\ u(0) = a, \ u(1) &= b. \end{aligned}$$

Let $k > 0$ and $a, b \geq 0$, then there exists $k_o > 0$, independent of a and b such that for $k \geq k_o$, (3_k) does not have a solution.

Proof. First of all, we want to show that there exists $\bar{k} > 0$ such that $(3_{\bar{k}})$ has no solution. Otherwise, we may suppose that (3_k) has a solution for arbitrary $k > 0$. So let k be an arbitrary positive real number and $u(t)$ be a solution of (3_k) corresponding to k. Since $u'' < 0$ on $[0,1]$ and if u attains its minimum on $(0,1)$ then it must be constant which is not a solution of (3_k), thus u attains its minimum at 0 and 1 and unique maximum on $(0,1)$. Let $u(t_o) = \max_{t\in[0,1]} u(t)$ for some $t_o \in (0,1)$ and denote $u_o = u(t_o)$. Multiplying both sides of (3_k) by $2u'$. We get

$$\{(u')^2\}' = -2ke^u u'$$

Integrating from t_o to t,

$$u'^2(t) = -2k\int_{t_o}^{t} e^{u(s)}u'(s)ds = -2k[e^{u(t)} - e^{u_o}].$$

Thus

$$u'(t) = \pm\sqrt{2k(e^{u_o} - e^{u(t)})}.$$

For $t \le t_o$, $u'(t) \ge 0$, thus $u'(t) = \sqrt{2k(e^{u_o} - e^{u(t)})}$ and

$$\frac{du(t)}{\sqrt{2k(e^{u_o} - e^{u(t)})}} = dt.$$

Integrating from 0 to t_o, we get

$$t_o = \int_a^{u_o} \frac{du}{\sqrt{2k(e^{u_o} - e^u)}}.$$

Similarly, for $t \ge t_o, u'(t) \le 0$ and we get

$$1 - t_o = \int_b^{u_o} \frac{du}{\sqrt{2k(e^{u_o} - e^u)}}.$$

Adding two equations

$$\begin{aligned}\sqrt{2k} &= \int_a^{u_o} \frac{du}{\sqrt{e^{u_o} - e^u}} + \int_b^{u_o} \frac{du}{\sqrt{e^{u_o} - e^u}} \\ &\le 2\int_0^{u_o} \frac{du}{\sqrt{e^{u_o} - e^u}} = 2\int_1^{e^{u_o}} \frac{dv}{v\sqrt{e^{u_o} - v}} \\ &= \frac{2}{\sqrt{e^{u_o}}}\int_{1/e^{u_o}}^{1} \frac{dw}{w\sqrt{1-w}} < \frac{4u_o}{\sqrt{e^{u_o}}}.\end{aligned}$$

Thus $k < 8u_o^2/e^{u_o}$. Since a function $8x^2/e^x$ is bounded for all $x \in \mathbf{R}^+$, the above inequality implies that k is bounded. This contradicts to the fact that k is arbitrary and thus there exists $\bar{k} > 0$ such that $(3_{\bar{k}})$ has no solution. Next we claim that if $(3_{\bar{k}})$ has no solution for some $\bar{k} > 0$, then (3_k) has no solution for all $k \geq \bar{k}$. Suppose that there exists $k^* > \bar{k}$ such that (3_{k^*}) has a solution then it is easy to check that $\beta(t)$, a solution of (3_{k^*}) and $\alpha(t)$ the straight line connecting $(0, a)$ and $(1, b)$ are an upper and a lower solution of $(3_{\bar{k}})$ respectively. Thus by Mawhin [3], $(3_{\bar{k}})$ has a solution $u(t)$ with $\alpha(t) \leq u(t) \leq \beta(t)$. This is a contradiction and the proof is done.

LEMMA 2. *Let (3_k) have a solution at $k = k^*$, then there exists $M > 0$, dependent of k^* such that for every possible solution u of (3_k) for $k^* \leq k$ one has $\|u\|_\infty < M$.*

PROOF. Assume that (3_{k^*}) has a solution for fixed k^*. Let k be a real number with $k \geq k^*$ and let u be a solution of (3_k). Then by the proof of Lemma 1, we get

$$\frac{e^{u_o}}{8u_o^2} < \frac{1}{k^*},$$

where $u_o = \max_{t \in [0,1]} u(t)$. The above inequality implies that u_o is bounded above by, we say, M which is dependent of k^* but independent of k. Thus we get the conclusion.

LEMMA 3. *Assume that (1_s) has a solution at $s = s^*$. Then there exists $R > 0$ such that for each s with $s \geq s^*$ and each possible solution u of (1_s), one has $\|u\|_\infty < R$.*

PROOF. Let s^* be fixed and $s \geq s^*$. We want to show the following fact first. If (1_s) has a solution $u(t)$, then the equation

$$\begin{aligned} &y''(t) + se^{y(t)} = 0, \\ &y(t_1) = u(t_1),\ y(1) = 0, \quad t_1 \in (0, 1) \end{aligned}$$

also has a solution. Indeed, let $u(t)$ be a solution of (1_s), then

$$u''(t) + se^{u(t)} = se^{u(t)}(1 - t^{-\frac{3}{2}}) \leq 0, \quad t \in [t_1, 1].$$

Thus $u(t)$ is an upper solution. On the other hand, let $\alpha(t)$ be the straight line connecting $(t_1, u(t_1))$ and $(1,0)$ then

$$\alpha''(t) + se^{\alpha(t)} > 0.$$

Thus $\alpha(t)$ is a lower solution. We also see that $\alpha(t) \leq u(t)$ for all $t \in [t_1, 1]$. Thus by Mawhin [3], the above equation has a solution. We now prove this lemma by contradiction. Suppose that there exists a sequence (s_n) with $s_n \geq s^*$ such that corresponding solution u_n of (1_{s_n}) satisfies $\|u_n\|_\infty \to \infty$. Without loss of generality, we may assume $u_n(\frac{1}{4}) \to \infty$ as $n \to \infty$. Consider

$$(4_n) \qquad \begin{aligned} & y''(t) + s_n e^{y(t)} = 0 \quad t \in [\frac{1}{4}, 1] \\ & y(\frac{1}{4}) = u_n(\frac{1}{4}),\ y(1) = 0. \end{aligned}$$

By the above argument, each (4_n) has a solution $y_n(t)$. Moreover $y_n(\frac{1}{4}) = u_n(\frac{1}{4}) \to \infty$ as $n \to \infty$. This implies that $\|y_n\|_\infty \to \infty$ as $n \to \infty$ and we get a contradiction to Lemma 2.

THEOREM 2. *There exists $s_o > 0$ such that (1_s) has at least one solution for $0 < s \leq s_o$ and no solution for $s > s_o$.*

PROOF. First, we will find an upper and a lower solution of (1_s) for certain s. Let $s \in (0, \frac{1}{e}]$ and consider

$$\begin{aligned} & u''(t) + t^{-\frac{3}{2}} = 0 \\ & u(0) = 0,\ u(1) = 0. \end{aligned}$$

The solution is $\beta(t) = 4(\sqrt{t} - t)$. We will claim that $\beta(t)$ is an upper solution of (1_s) for $0 < s \leq \frac{1}{e}$. Since $0 \leq \beta(t) \leq 1$ for any $t \in [0,1]$ and $s \leq \frac{1}{e}$

$$se^{\beta(t)} \leq se \leq 1.$$

Thus

$$\beta''(t) + st^{-\frac{3}{2}} e^{\beta(t)} = t^{-\frac{3}{2}}(se^{\beta(t)} - 1) \leq 0.$$

This shows that $\beta(t)$ is the upper solution of (1_s). $\alpha(t) \equiv 0$ is obviously the lower solution of (1_s) and $\alpha(t) \leq \beta(t)$ for any $t \in [0,1]$. Thus, by Theorem 1, (1_s) has a solution for $0 < s \leq \frac{1}{e}$. Second, for fixed $s_1 > 0$ assume that (1_{s_1}) has a solution then for arbitrary $s \in (0, s_1)$, (1_s) also has a solution. Indeed, let $\beta(t)$ be a solution of (1_{s_1}), then

$$\beta''(t) + st^{-\frac{3}{2}}e^{\beta(t)} = (s - s_1)t^{-\frac{3}{2}}e^{\beta(t)} < 0.$$

Thus $\beta(t)$ is the upper solution of (1_s) and obviously $\alpha(t) \equiv 0$ is the lower solution of (1_s). Therefore (1_s) has at least one solution for $s \in (0, s_1)$. Let $s_o = \sup\{s > 0 : (1_s) \text{ has at least one solution.}\}$, then we may notice by the above argument that $s_o \geq \frac{1}{e}$ with the possibility $s_o = \infty$. Third, we claim that $s_o < \infty$. We will show by contradiction. Assume that there exists a sequence of parameters (s_n) with $s_n \to \infty$ such that each (1_{s_n}) has a solution u_n. On the interval $[\frac{1}{4}, 1]$, consider equation (4_n) again. By the same argument as in the proof of Lemma 3, (4_n) has a solution for all n. Since $s_n \to \infty$, the above conclusion contradicts to Lemma 1. Consequently $0 < s_o < \infty$. Finally, we show the existence of solution for (1_s) at $s = s_o$. By the definition of s_o, we may choose an increasing sequence (s_n) such that $s_n \to s_o$ and each (1_{s_n}) has a solution. Let u_n be a solution of (1_{s_n}) then by Lemma 3 and Arzela-Ascoli Theorem, (u_n) has a subsequence converging to $u \in C[0,1]$. We claim that u is a solution of (1_{s_o}). Now u is a solution of (1_s) if and only if u satisfies

$$u(t) = s\int_0^1 G(t,\tau)\tau^{-\frac{3}{2}}e^{u(\tau)}d\tau,$$

where

$$G(t,\tau) = \begin{cases} \tau(1-t) & \text{for } 0 \leq \tau \leq t \\ t(1-\tau) & \text{for } t \leq \tau \leq 1. \end{cases}$$

Since u_n is a solution of (1_{s_n}),

$$u_n(t) = s_n\int_0^1 G(t,\tau)\tau^{-\frac{3}{2}}e^{u_n(\tau)}d\tau.$$

By Lemma 3, there exists a constant $R > 0$ such that $\|u_n\|_\infty < R$, for all n. Thus

$$|G(t,\tau)\tau^{-\frac{3}{2}}e^{u_n(\tau)}| \leq e^R|G(t,\tau)|\tau^{-\frac{3}{2}} \leq e^R\tau(1-\tau)\tau^{-\frac{3}{2}}.$$

Let $h(\tau) = e^R(1-\tau)\tau^{-\frac{1}{2}}$, then $h \in L^1[0,1]$ and applying Lebesgue Dominated Convergence Theorem, we get

$$\begin{aligned} u(t) = \lim_{n\to\infty} u_n(t) &= s_o \int_0^1 \lim_{n\to\infty} G(t,\tau)\tau^{-\frac{3}{2}} e^{u_n(\tau)} d\tau \\ &= s_o \int_0^1 G(t,\tau)\tau^{-\frac{3}{2}} e^{u(\tau)} d\tau. \end{aligned}$$

Therefore u is a solution of (1_{s_o}) and this completes the proof.

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