Comm. Korean Math. Soc. **10** (1995), No. 3, pp. 707–714

A FUNCTIONAL CENTRAL LIMIT THEOREM FOR POSITIVELY DEPENDENT RANDOM VECTORS

TAE-SUNG KIM AND JONG-IL BAEK

ABSTRACT. In this note,we extend the concepts of linearly positive quadrant dependence to the random vectors and prove a functional central limit theorem for positively quadrant dependent sequence of R^d-valued or separable Hilbert space valued random elements which satisfy a covariance summability condition. This result is an extension of a functional central limit theorem for weakly associated random vectors of Burton et al. to positive quadrant dependence case.

1. Introduction

Many recent papers have been concerned with concepts of positive dependence for families of random variables (for example see Block and Ting(1981), Shaked(1982) and the references there in). Lehmann(1966) introduced the notion of positive quadrant dependence: random variables X_1 and X_2 are called positive quadrant dependence if for all real r_1, r_2

$$P\{X_1 > r_1, X_2 > r_2\} \geq P\{X_1 > r_1\}P\{X_2 > r_2\}. \tag{1.1}$$

This definition was subsequently extended to multivariate case in Esary, Proschan and Walkup(1967): a family $\{X_1, \cdots, X_m\}$ of random variables is said to be associated if

$$Cov(f(X_1, \cdots, X_m), g(X_1, \cdots, X_m)) \geq 0 \tag{1.2}$$

for any coordinatewise nondecreasing functions f, g on R^m such that the covariance exists. Infinite families are associated if every finite subfamily

Received February 11, 1995. Revised June 16, 1995.

1991 AMS Subject Classification: Primary 60F17 ; secondary 62H05.

Key words and phrases: association,linearly positive quadrant dependence, functional central limit theorem.

is associated. Newman(1980) was the first who showed that for positively dependent sequences approximate uncorrelatedness implies approximate independence such that useful limit theorems can be obtained. In the following years several extensions and generalizations of these results were given. There exist (functional) central limit theorems [8, 10] and laws of the iterated logarithm [5] ,as well as laws of large number[1] for associated sequences. Most of these results,however,can not be applied to weaker concepts of positive dependence. For pairwise positively quadrant dependent sequences only a strong law of large numbers is known [1]. In stead of association Newman's original central limit theorem requires only that positive linear combinations of the random variables are positively quadrant dependent. The definition of positive dependence introduced by Newman is the following: a sequence $\{X_j : j \geq 1\}$ of random variables is said to be linearly positive quadrant dependent if for any disjoint A,B and positive $r_j's$ $\sum_{i \in A} r_i X_i$and $\sum_{j \in B} r_j X_j$ are positively quadrant dependent. Burton,R.M, Dabrowski,A and Dehling, H [4] extended the notion of association to random vectors: let $(X_1, X_2, \cdots, X_m)$ be joint R^d-valued random vectors. They are said to be associated if for any coordinatewise increasing functions f, g on R^{md}

$$Cov(f(X_1, \cdots, X_m), g(X_1, \cdots, X_m)) \geq 0$$

whenever the covariance is defined. An infinitely family of R^d-valued random vectors is associated if every finite subfamily is associated. The functional central limit theorem due to Burton,Dabrowski and Dehling [4] is

THEOREM A. *Let*$\{X_i : i \geq 1\}$ *be a strictly stationary associated sequence of* R^d*-valued random vectors,centered at expectations and with* $E\|X_1\|^2 < \infty$. *Define for* $t \in [0,1], n \geq 1$ $W_n(t) = n^{-1/2} \sum_{j \leq [nt]} X_j$. *If*

$$(1.3) \qquad E\|X_1\|^2 + 2\sum_{i=2}^{\infty} \sum_{j=1}^{d} E\left(X_1^{(j)} X_i^{(j)}\right) = \sigma^2 < \infty$$

then as $n \to \infty$ $W_n \xrightarrow{w} B^d$ *where* $\xrightarrow{w}$ *indicates weak convergence, and* B^d *is a d-dimensional Wiener process with covariance matrix* $\Gamma = [\sigma_{ki}]$,

$$(1.4) \qquad \sigma_{kj} = E(X_1^{(k)} X_1^{(j)}) + \sum_{j=2}^{\infty} \left[E(X_1^{(k)} X_i^{(j)}) + E(X_1^{(j)} X_i^{(k)})\right]$$

In this note, we give a definition of linearly positive quadrant dependence for random vectors with values in R^d [or any Banach lattice] and extend Theorem A to linearly positive quadrant dependent case. These results are extended to random vectors with values in a separable Hilbert space.

2. Results

DEFINITION 2.1. A sequence $\{X_j : j \geq 1\}$ of R^d-valued random vectors is said to be linearly positive quadrant dependent if for any disjoint A,B and positive vector r_j's $\sum_{i\in A} r_i X_i$ and $\sum_{j\in B} r_j X_j$ are positively quadrant dependent.

THEOREM 2.2. *Let $\{X_j : j \geq 1\}$ be a strictly stationary linearly positive quadrant dependent sequence of R^d-valued random vectors, centered at expectations and with $E\|X_1\|^2 < \infty$. Assume that (1.3) holds. Then $\{X_j : j \geq 1\}$ fulfills the functional central limit theorem.*

DEFINITION 2.3. Let $\{X_j : j \geq 1\}$ be a sequence of random variables taking values in a separable Hilbert space $(H, <.,.>)$. $\{X_j : j \geq 1\}$ is called linearly positive quadrant dependent, if for some orthonormal basis $(e_k, k \geq 1)$ of H and for any $d \geq 1$ the d-dimensional sequence $(<X_i, e_1>, \cdots, <X_i, e_d>), i \geq 1$ is linearly positive quadrant dependent.

THEOREM 2.4. *Let $\{X_j : j \geq 1\}$ be a strictly stationary linearly positive quadrant dependent sequence of H-valued random variables with $EX_1 = 0$, and $E\|X_1\|^2 < \infty$. Define*

$$W_n(t) = \begin{cases} n^{-1/2} \sum_{j \leq k} X_j & \text{if } t = k/n\ , \\ \text{linear in between.} \end{cases}$$

Assume

$$\sigma^2 = E\| X_1 \|^2 + \sum_{j=2}^{\infty} E < X_1, X_j > < \infty. \tag{2.1}$$

Then $W_n \xrightarrow{w} W$, where W is a Wiener process on H with covariance structure $\Gamma(f,g) = E(<f, X_1><g, X_1>) + \sum_{j=2}^{\infty}\{E(<f, X_1><g, X_j>) + E(<g, X_1><f, X_j>)\}$

3. Proofs

LEMMA 3.1. (Burton, Darbrowski, Dehling (1986)) *Let* $Y_1, Y_2, \cdots, Y_k$ *and* $Y_1', Y_2' \cdots, Y_k'$ *be two sets of* R^d *-valued random vectors having finite moment generating functions. Suppose for all nonnegative vectors* $a_1, a_2, \cdots, a_n$ *the joint distributions of* $(< a_1, Y_1 >, \cdots, < a_k, Y_k >)$ *and* $(< a_1, Y_1' >, \cdots, < a_k, Y_k' >)$ *coincide. Then* $(Y_1, \cdots, Y_k)$ *and* $(Y_1', \cdots, Y_k')$ *have the same distribution.*

LEMMA 3.2. *Let* $\{X_j : j \geq 1\}$ *be a sequence of linearly positive quadrant dependent random variables with* $EX_j = 0, EX_j^2 < \infty$. *Then*

$$E\{(\max_{1 \leq i \leq m} S_i)^2\} \leq E(S_m^2) \tag{3.1}$$

where $S_m = X_1 + \cdots + Xm$.

PROOF. Put

$$K_m = \min(X_1 + \cdots + X_m, \cdots X_2 + \cdots + X_m \cdots, X_m, 0)$$
$$L_m = \max(X_2, \cdots, X_2 + X_3, \cdots, X_2 + \cdots + X_m)$$
$$J_m = \max(0, L_m).$$

Then we have $Cov(X_1, K_m) \geq 0$ since X_i's are linearly positive quadrant dependent, $J_m^2 \leq L_m^2$ pointwise and $\max_{1 \leq i \leq m} S_i = X_1 + J_m$. Thus

$$\begin{aligned} &E\{(\max_{1 \leq i \leq m} S_i)^2\} \\ =&E\{(X_1 + J_m)^2\} \\ =&EX_1^2 + 2Cov(X_1, J_m) + E(J_m^2) \\ =&EX_1^2 + 2Cov(X_1, X_2 + \cdots + X_m) - 2Cov(X_1, K_m) \\ \leq&EX_1^2 + 2Cov(X_1, X_2 + \cdots + X_m) + E(L_m^2). \end{aligned} \tag{3.2}$$

The proof is completed by induction on m since the induction hypothesis implies $E(L_m^2) \leq E(X_2 + \cdots + X_m)^2$ which together with (3.2) yields (3.1).

LEMMA 3.3. *Let $\{X_j : j \geq 1\}$ be a sequence of a strictly stationary and linearly positive quadrant dependent random variables with $EX_i = 0, EX_j^2 < \infty$. Define $S_m^* = \max(0, S_1, \cdots, S_m)$. Then for $\lambda_1 < \lambda_2$,*

$$(3.3) \qquad P(S_m^* \geq \lambda_2) \leq \left(1 - \sigma_m^2/(\lambda_2 - \lambda_1)^2\right)^{-1} P(S_m \geq \lambda_1)$$

$$(3.4) \qquad P(\max(|S_1|, \cdots, |S_m|) \geq \lambda\sigma_m) \leq 2P(|S_m| \geq (\lambda - \sqrt{2})\sigma_m)$$

where $\sigma_m^2 = E(S_m^2)$

PROOF. We now prove this lemma along the lines of the proof of Theorem 3 of [10]. For completeness we repeat some of the argument given in that paper for $\lambda_1 < \lambda_2$

$$\begin{aligned} P(S_m^* > \lambda_2) &\leq P(S_m \geq \lambda_1) + P(S_{m-1}^* \geq \lambda_2, S_{m-1}^* - S_m > \lambda_2 - \lambda_1) \\ &\leq P(S_m \geq \lambda_1) + P(S_{m-1}^* > \lambda_2) P(S_{m-1}^* - S_m > \lambda_2 - \lambda_1) \\ &\leq P(S_m \geq \lambda_1) + P(S_m^* \geq \lambda_2) E((S_{m-1}^* - S_m)^2)/(\lambda_2 - \lambda_1)^2 \end{aligned}$$

where the second inequality follows from the fact that S_{m-1}^* and $S_m - S_{m-1}^*$ are positively quadrant dependent since the $X's$ are linearly positive quadrant dependent. Now Lemma 3.2 with X_i replaced by $Y_i = -X_{m-i+1}$ yields that

$$\begin{aligned} &E\{(S_{m-1}^* - S_m)^2\} \\ =&E\left\{[\max(Y_1, Y_1 + Y_2, \cdots, Y_1 + \cdots + Y_m)]^2\right\} \leq E(S_m^2) \end{aligned}$$

and thus we obtian, for $(\lambda_2 - \lambda_1)^2 \geq E(S_m^2) = \sigma_m^2$, (3.3). By adding to (3.3) the analogous inequality with each X_i replaced by $-X_i$,and by choosing $\lambda_2 = \lambda\sigma_m, \lambda_1 = (\lambda - \sqrt{2})\sigma_m$,we also obtain (3.4)

PROOF OF THEOREM 2.2. Tightness of the sequence $\{W_n(.), n \geq 1\}$ follows from Lemma 3.3. Hence, it remains to show that the only possible limit point is Brownian motion in R^d with covariance structure $\sum = (\sigma_{kl})$, Let $Z(.)$ be a limit point of $(W_n(.))$. We have to prove

(i) $Z(t + h) - Z(t)$ has normal distribution with covariance,

(ii) Z has independent increments.

Let $a \in R^d$ be nonnegative vector. Then $< a, (Z(t+h) - Z(t)) >$ has, by the Newman and Wright result, a normal distribution with variance a $\sum a'$. Now we can apply Lemma 3.1 with k=1 which yields (i).
Let $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq 1$ be given. Then

$$(W_{n'}(t_1), W_{n'}(t_2) - W_{n'}(t_1), \cdots, W_{n'}(t_k) - W_{n'}(t_{k-1})) \qquad n' \in N$$

converges in distribution to

$$(Z(t_1), Z(t_2) - Z(t_1), \cdots, Z(t_k) - Z(t_{k-1})).$$

Let $a_1, \cdots, a_k \in R^d$ be nonnegative vectors. Then

$$\big(< a_1, W_{n'}(t_1) >, < a_2, W_{n'}(t_2) - W_{n'}(t_1) >, \\ \cdots, < a_k, W_{n'}(t_k) - W_{n'}(t_{k-1}) >\big)$$

converges in distribution to

$$(< a_1, Z(t_1) >, \cdots, < a_k, (Z(t_k) - Z(t_{k-1})) >).$$

The coordinates of the last vector are hence pairwise positively quadrant dependent according to Lemma 4 of Birkel [2] and by a simple computation involving (2.1) also uncorrelated, which together imply independence by Lemma 2 of Birkel [2]. Now we again apply Lemma 3.1 to obtain the independence of the increments of the Z process.

PROOF OF THEOREM 2.4. Let (e_k) be the orthonormal basis with respect to which the sequence $\{X_j : j \geq 1\}$ is linearly positive quadrant dependent. For $N \geq 1$ let P_N be the projection onto the subspace generated by $e_1, \cdots, e_N$ and $Q_N = 1 - P_N$. It is easy to see that, for any $k \geq 1$,

$$E\Big(n^{-1/2} \sum_{j \leq n} (< X_j, e_k >)^2\Big) \uparrow \sigma_k^2$$

where $\sigma_k^2 = \Gamma(e_k, e_k)$.

Hence for any $\varepsilon > 0$ there exist N_0, n_0 such that

$$E \parallel n^{-1/2} Q_N \sum_{j \le n} X_j \parallel^2 \le \varepsilon \text{ for every } n \ge n_0, N \ge N_0.$$

From Theorem 2.2 we have that $P_N W_n$ converges in distribution to $P_N W$. Hence it remains to show that $Q_N W_n$ (and $Q_N W$) become small as $N \to \infty$, respectively. By Lemma 3.2 we have

$$E\Big\{\Big(\max_{m \le n}\Big[\sum_{j=1}^{m} < X_i, e_k >\Big]\Big)^2\Big\} \le 2E\Big\{\Big(\sum_{j=1}^{n} < X_j, e_k >\Big)^2\Big\}$$

and hence

$$\begin{aligned}
&E\Big(n^{-1} \max_{m \le n} \parallel \sum_{j=1}^{m} Q_N X_j \parallel^2\Big) \\
&\le (n^{-1}) \sum_{k=N+1}^{\infty} E\Big(\max_{m \le n} \sum_{i=1}^{m} < X_i, e_k >\Big)^2 \\
&\le 2(n^{-1}) \sum_{k=N+1}^{\infty} E\Big(\sum_{i=1}^{n}\Big(< X_i, e_k >\Big)^2\Big) \\
&\le 2(n^{-1}) E \parallel Q_N \sum_{i=1}^{n} X_i \parallel^2 \\
&\le \varepsilon.
\end{aligned}$$

A similar estimate holds for $Q_N W$ and this completeness the proof of Theorem 2.4.

References

1. Birkel, T., *A note on the strong law of large numbers for positively dependent random variables*, Statist. and Probab. Letters **7** (1989), 17-20.
2. Birkel, T., *A functional central limit theorem for positively dependent random variables*, J. Multivariate Anal. **44** (1993), 314-320.
3. Block, H. W. and Ting, M. L., *Some concepts of multivariate dependence*, Comm. Statist. A **10** (1981), 749-762.
4. Burton, R. M, Dabrowski, A. R. and Dehling, H., *An invariance principle for weakly associated random vectors*, Stoc. Proc. and Appl. **23** (1986), 301-306.

5. Dabrowski, A. R., *A functional law of the iterated logarithm for associated sequences*, Statistics and Probab.Letters **3** (1985), 209-212.
6. Esary, I. Proschan, F. and Walkup, D, *Association of random variables with applications*, Ann. Math. Statist. **38** (1967), 1466-1474.
7. Lehmann, E. L., *Some concepts of dependence*, Ann. Math. Statist **37** (1966), 1137-1153.
8. Newman, C. M., *Normal fluctuations and the FKG inequalities*, Comm. Math. Phys. **79** (1980), 119-128.
9. Newman, *Asymptotic independence and limit theorems for positively and negatively dependent random variables*, in. Tong, H. ed., IMS Lecture Notes-Monograph series **5** (1984), 127-140.
10. Newman, C. M. and Wright, A. L., *An invariance principle for certain dependent sequences*, Ann. Probab. 9 (1981), 671-675.
11. Shaked, M, *A general theory of some positive dependence notions*, J. Multivariate Anal. **12** (1982), 199-128.

Department of Statistics
Won-Kwang University
Iksan, Korea 570-749