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ON THE FIXED-POINT THEOREMS ON THE INFRASOLVMANIFOLDS

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ABSTRACT. Fixed-point theory has an extension to coincidences. For a pair of maps $f, g: X_1 \to X_2$, a coincidence of f and g is a point $x \in X_1$ such that $f(x) = g(x)$, and $\mathrm{Coin}(f, g) = \{ x \in X_1 \mid f(x) = g(x) \}$ is the coincidence set of f and g. The Nielsen coincidence number $N(f, g)$ and the Lefschetz coincidence number $L(f, g)$ are used to estimate the cardinality of $\mathrm{Coin}(f, g)$. The aspherical manifolds whose fundamental group has a normal solvable subgroup of finite index is called infrasolvmanifolds. We show that if M_1, M_2 are compact connected orientable infrasolvmanifolds, then $N(f, g) \geq |L(f, g)|$ for every $f, g: M_1 \to M_2$.

1. Introduction

For a continuous self-map $f: X \to X$, the fixed point set $\{ x \in X \mid f(x) = x \}$ is denoted by $\mathrm{Fix}(f)$. The Nielsen number $N(f)$ provides a lower bound on the number of fixed points of g, for all maps g homotopic to f. However, it is very difficult to compute $N(f)$ from its definition. The Lefschetz number $L(f)$ is computable invariant. We search to find conditions on either the space X or the map f which allow $N(f)$ and $L(f)$ to be related.

Anosov showed that $N(f) = |L(f)|$ for all maps on compact nilmanifolds ([1]). On the other hand, the counter-examples on Klein bottle $(N(f) = 4, L(f) = 2)$ show that the equality does not hold for all maps on solvmanifolds, nor on infranilmanifolds. C. McCord showed that $N(f) \geq |L(f)|$ for all maps on compact solvmanifolds ([11], Theorem 2.5).

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Fixed-point theory has an extension to coincidences. The Nielsen coincidence number $N(f,g)$ and the Lefschetz coincidence number $L(f,g)$ (For the precise definitions see §2 of [13]) are the generalization of $N(f)$ and $L(f)$ respectively. Brooks and Wang showed that $N(f,g) = |L(f,g)|$ when $M_1 = M_2$ is an infranilmanifold ([4]). C. McCord showed that $N(f,g) \geq |L(f,g)|$ when M_1, M_2 are compact orientable solvmanifolds of the same dimension, with equality if M_2 is a nilmanifold ([12], Theorem 2). Nilmanifolds, infranilmanifolds and solvmanifolds are the subclasses of the infrasolvmanifolds (For the precise definitions see §6 of [13], §4 of [14] and p. 546 of [8]). Every infrasolvmanifold has a finite regular cover by a solvmanifold and every infranilmanifold has a finite regular cover by a nilmanifold. In all cases, the universal cover is contractible, so the manifolds are aspherical.

In this paper, we use the liftings of f and g which C. McCord used in [13], and we show that if M_1, M_2 are compact connected orientable infrasolvmanifolds of the same dimension, then $N(f,g) \geq |L(f,g)|$ for every $f, g\colon M_1 \to M_2$, with equality for every f and g if M_2 is a nilmanifold.

2. Brief review of coincidence theory and lifts

For a pair of maps $f, g\colon M_1 \to M_2$, we denote the coincidence set of f and g by

$$\mathrm{Coin}(f,g) = \{\, x \in X_1 \mid f(x) = g(x) \,\}.$$

For some $x, y \in \mathrm{Coin}(f,g)$, if there exists a path ω in X_1 from x to y with $f \cdot \omega \simeq g \cdot \omega$ (rel $\{0,1\}$), then set $x \sim y$. This relation classify $\mathrm{Coin}(f,g)$ into coincidence classes. Each class is compact and open in $\mathrm{Coin}(f,g)$ ([2], p. 22 and [13], §2). The set of the classes of $\mathrm{Coin}(f,g)$ is denoted by $\mathcal{R}(f,g)$.

If $F\colon f_0 \simeq f_1$ and $G\colon g_0 \simeq g_1$, then coincidence classes $S_0 \in \mathcal{R}(f_0, g_0)$ and $S_1 \in \mathcal{R}(f_1, g_1)$ are <u>(F,G)-related</u> if there exist $x_0 \in S_0$, $x_1 \in S_1$ and a path ω in X_1 such that the paths $< F,\omega >$ and $< G,\omega >$, defined by $< F,\omega > (t) = F_t(\omega(t))$, are homotopic in X_2. A class $S \in \mathcal{R}(f,g)$ is <u>topologically essential</u> if, for every $F\colon f \simeq f'$, $G\colon g \simeq g'$, there exists a class $S' \in \mathcal{R}(f', g')$ which is (F,G)-related to S. We denote the set of essential classes by $\mathcal{E}(f,g)$. The <u>Nielsen coincidence number</u> $N(f,g)$ is the $|\mathcal{E}(f,g)|$ (cardinality of $\mathcal{E}(f,g)$).

Suppose M_1 and M_2 are both compact connected orientable n-manifolds. For each coincidence class S, a coincidence index $\operatorname{Ind}(f, g, S)$ is defined ([13], p. 348). We have

$$\operatorname{Ind}(f,g) = \sum \operatorname{Ind}(f,g,S),$$

the sum of the coincidence index over all coincidence classes.

In rational coefficients, let $D_i: H^{n-p}(M_i) \longrightarrow H_p(M_i)$ be the Poincaré duality isomorphism and let $\theta_p(f,g)$ be the composition

$$H_p(M_1) \xrightarrow{f_*} H_p(M_2) \xrightarrow{D_2^{-1}} H^{n-p}(M_2) \xrightarrow{g^*} H^{n-p}(M_1) \xrightarrow{D_1} H_p(M_1)$$

([15], p. 176).

The Lefschetz coincidence number $L(f,g)$ is defined as

$$\sum_{p=0}^{n} (-1)^p \operatorname{tr} \theta_p(f,g),$$

and the Lefschetz coincidence theorem states that $L(f,g) = \operatorname{Ind}(f,g)$. $L(f,g)$ is the only defined for orientable manifolds because the coincidence index is only defined in that setting.

We now briefly review the covering spaces and lifts ([13], [7]). Fix base points $x_1 \in X_1$, $x_2 \in X_2$, and assume that $f(x_1) = x_2 = g(x_1)$. Let π_i denote $\pi(X_i, x_i)$ and define

$$\mathcal{C}(\pi_i) = \{\, \Gamma \lhd \pi_i \mid [\pi_i : \Gamma] < \infty \,\}$$

($[\pi_i : \Gamma] = |\pi_i/\Gamma|$). There is a one-to-one correspondence between elements of $\mathcal{C}(\pi_i)$ and finite regular covers of X_i. Recall that any manifold has an orientable cover.

For a pair of maps $f, g: X_1 \to X_2$, fix $\Gamma_2 \in \mathcal{C}(\pi_2)$ and corresponding finite regular cover $p_2: \tilde{X}_2 \to X_2$. Given a cover $p_1: \tilde{X}_1 \to X_1$ and corresponding $\Gamma_1 \in \mathcal{C}(\pi_1)$, f and g lift to some $\tilde{f}, \tilde{g}: \tilde{X}_1 \to \tilde{X}_2$ if and only if $f_\#, g_\#: \pi_1 \to \pi_2$ have $f_\#(\Gamma_1), g_\#(\Gamma_1) \subseteq \Gamma_2$. So define

$$\mathcal{C}(f,g,\Gamma_2) = \{\, \Gamma_1 \in \mathcal{C}(\pi_1) \mid f_\#(\Gamma_1), g_\#(\Gamma_1) \subseteq \Gamma_2 \,\}.$$

The lifting diagram

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\tilde{f},\tilde{g}} & \tilde{X}_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ X_1 & \xrightarrow[f,g]{} & X_2 \end{array}$$

is called the $\Gamma_1 - \Gamma_2$ lifting diagram of f and g. For any lifting diagram, the lifts $\tilde{f}, \tilde{g}$ may not have a coincidence index defined. We therefore define for every $\Gamma_2 \in \mathcal{C}(\pi_2)$ the set

$$\mathcal{IC}(f, g, \Gamma_2) = \{\, \Gamma_1 \in \mathcal{C}(\pi_1) \mid f_\#(\Gamma_1), g_\#(\Gamma_1) \subseteq \Gamma_2 \text{ and an index is defined for the } \Gamma_1 - \Gamma_2 \text{ lifts} \,\}.$$

In any $\Gamma_1 - \Gamma_2$ lifting diagram, Γ_i has covering group $\Phi_i = \pi_i/\Gamma_i$. $f_\#$ and $g_\#$ induce maps $\bar{f}, \bar{g}\colon \Phi_1 \to \Phi_2$. If $S \in \mathcal{R}(f, g)$, then define

$$C_\#(f, g, S) = \{\, \alpha \in \pi_1 \mid f_\#(\alpha) = g_\#(\alpha) \,\},$$

where $f_\#$ and $g_\#$ are based at some $x \in S$. $C_\#(f, g, S)$ is a subgroup of π_1, but is not necessarily normal.

3. Main Results

THEOREM 1. ([13], Cor. 5.7) *Suppose M_1, M_2 are compact orientable manifolds of the same dimension, and neither is a surface with negative Euler characteristic. If $S \in \mathcal{R}(f, g)$ and Θ is the set of coincidence classes covering S in the $\Gamma_1 - \Gamma_2$ lifting diagram, then*

$$|\phi_1| Ind(f, g, S) = \sum_{\tilde{S} \in \Theta} Ind(\beta \circ \tilde{f}, \tilde{g}).$$

We now introduce two more concepts for the $\Gamma_1 - \Gamma_2$ lifting diagram of $f, g\colon M_1 \to M_2$. Let $\Gamma_2 \in \mathcal{C}(\pi_2)$, $\Gamma_1 \in \mathcal{C}(f, g, \Gamma_2)$ and lifts $\tilde{f}, \tilde{g}$ of f and g. We define a Nielsen-type coincidence number

$$\tilde{N}(f, g, \Gamma_1) = \frac{1}{|\phi_1|} \sum_{\beta \in \phi_2} N(\beta \circ \tilde{f}, \tilde{g}).$$

If $\Gamma_1 \in \mathcal{IC}(f, g, \Gamma_2)$, we define a Lefschetz-type coincidence number

$$\tilde{L}(f, g, \Gamma_2) = \frac{1}{|\phi_1|} \sum_{\beta \in \phi_2} |L(\beta \circ \tilde{f}, \tilde{g})|.$$

By simply combining two corollaries 7.6 and 5.10 of [13], we list the following theorem.

THEOREM 2. (Cor. 7.6 and Cor. 5.10 of [13]) *If M_1, M_2 are compact connected orientable infrasolvmanifolds of the same dimension, then $C_{\#}(f, g, S) \subset \Gamma_1$ for every $\Gamma_1 \in \mathcal{C}(\pi_1)$ and every $S \in \mathcal{E}(f, g)$. Therefore $\tilde{N}(f, g, \Gamma_1) = N(f, g)$ for every $\Gamma_1 \in \mathcal{C}(f, g, \Gamma_2)$.*

We are now concerned with the main results.

THEOREM 3. *Suppose M_1, M_2 are compact connected orientable manifolds of the same dimension and neither is a surface with negative Euler characteristic. If $\Gamma_1 \in \mathcal{IC}(f, g, \Gamma_2)$, then*

$$L(f, g) = \frac{1}{|\phi_1|} \sum_{\beta \in \phi_2} L(\beta \circ \tilde{f}, \tilde{g}).$$

PROOF. By Using the property of coincidence index and Theorem 1, we have

$$\begin{aligned} L(\beta \circ \tilde{f}, \tilde{g}) &= \sum_{\tilde{S} \in \mathcal{R}(\beta \circ \tilde{f}, \tilde{g})} \operatorname{Ind}(\beta \circ \tilde{f}, \tilde{g}, \tilde{S}) \\ &= |\phi_1| \sum_{S \subset p_1(\operatorname{Coin}(\beta \circ \tilde{f}, \tilde{g}))} \operatorname{Ind}(f, g, S). \end{aligned}$$

The sum of the both sides of this equality over all $\beta \in \Phi_2$ derives the result :

$$\begin{aligned} \sum_{\beta \in \phi_2} L(\beta \circ \tilde{f}, \tilde{g}) &= |\phi_1| \sum_{\beta \in \phi_2} \sum_{S \subset p_1(\operatorname{Coin}(\beta \circ \tilde{f}, \tilde{g}))} \operatorname{Ind}(f, g, S) \\ &= |\phi_1| \sum_{S \in \mathcal{R}(f, g)} \operatorname{Ind}(f, g, S) \\ &= |\phi_1| \cdot L(f, g). \qquad \square \end{aligned}$$

THEOREM 4. (Thm. 2 of [12]) *If M_1, M_2 are compact connected orientable solvmanifolds of the same dimension, then $N(f,g) \geq |L(f,g)|$ for all $f, g: M_1 \to M_2$. Moreover, if M_2 is a nilmanifold, then $N(f,g) = |L(f,g)|$ for every (f,g).*

THEOREM 5. *If M_1, M_2 are compact connected orientable infrasolvmanifolds of the same dimension, then $N(f,g) \geq |L(f,g)|$ for every $f, g: M_1 \to M_2$. Moreover, if M_2 is a nilmanifold, then $N(f,g) = |L(f,g)|$ for every (f,g).*

PROOF. Choose a solvable $\Gamma_2 \in \mathcal{C}(\pi_2)$ and a solvable $\Gamma_1 \in \mathcal{IC}(f,g,\Gamma_2)$. Then by the definition of $\tilde{N}(f,g,\Gamma_1)$ and Theorem 2, we have :

$$N(f,g) = \frac{1}{|\phi_1|} \sum_{\beta \in \phi_2} N(\beta \circ \tilde{f}, \tilde{g}).$$

In the $\Gamma_1 - \Gamma_2$ lifting diagram, $\tilde{f}, \tilde{g}: \tilde{M}_1 \to \tilde{M}_2$ satisfies the conditions of Theorem 4. Thus we have:

$$\begin{aligned} \frac{1}{|\phi_1|} \sum_{\beta \in \phi_2} N(\beta \circ \tilde{f}, \tilde{g}) &\geq \frac{1}{|\phi_1|} \sum_{\beta \in \phi_2} |L(\beta \circ \tilde{f}, \tilde{g})| \\ &\geq \frac{1}{|\phi_1|} \Big| \sum_{\beta \in \phi_2} L(\beta \circ \tilde{f}, \tilde{g}) \Big| \\ &= |L(f,g)| \qquad \text{(by Theorem 3).} \end{aligned}$$

If M_2 is a nilmanifold, choose a solvable $\Gamma_1 \in \mathcal{IC}(f,g,\pi_2)$. Then f, g have unique liftings $\tilde{f} = f \circ p_1, \tilde{g} = g \circ p_1: \tilde{M}_1 \to M_2$, and the covering map $p_1: \tilde{M}_1 \to M_1$ has $\deg(p_1) = |\phi_1|$ and

$$L(\tilde{f}, \tilde{g}) = \deg(p_1) \cdot L(f,g) = |\phi_1| \cdot L(f,g).$$

By the moreover part of the Theorem 4,

$$\text{(A)} \qquad N(\tilde{f}, \tilde{g}) = |L(\tilde{f}, \tilde{g})|.$$

On the other hand, since $\Gamma_2 = \pi_2, \phi_2 = 1$. Thus we have

$$\begin{aligned} N(f,g) &= \frac{1}{|\phi_1|} N(\tilde{f},\tilde{g}) \\ &= \frac{1}{|\phi_1|} N(f \circ p_1, g \circ p_1) \\ &= \frac{1}{|\phi_1|} |L(f \circ p_1, g \circ p_1)| \qquad \text{(by (A))} \\ &= \frac{1}{|\phi_1|} \cdot |\phi_1||L(f,g)| = |L(f,g)| \end{aligned}$$

□

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