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MULTIPLIERS ON THE DIRICHLET SPACE $D(\Omega)$

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ABSTRACT. Recently S.Axler proved that every sequence in the unit disk U converging to the boundary contains an interpolating subsequence for the multipliers of the Dirichlet space $D(U)$. In this paper we generalizes Axler's result to the finitely connected planer domains such that the Dirichlet spaces are contained in the Bergman spaces.

1. Notations and terminologies

Througout this paper Ω denotes a domain in the complex plane **C** such that no connected component of $\partial\Omega$ is equal to a point. The *Bergman space* $B(\Omega)$ is the Hilbert space of analytic functions f on Ω such that $\int_\Omega |f|^2\, dA < \infty$, with the inner product

$$< f, g >_{B(\Omega)} = \int_\Omega f\overline{g}dA$$

where dA denotes the usual area measure on Ω. Let z_0 be in Ω. The *Dirichlet space* $D(\Omega, z_0)$ is the Hilbert space of analytic functions f on Ω such that $\int_\Omega |f'|^2\, dA < \infty$ and $f(z_0) = 0$, with the inner product

$$< f, g >_{D(\Omega)} = \int_\Omega f'\overline{g'}dA. \tag{1.1}$$

Changing the distinguished point z_0 gives a space that is obtained from the original by subtracting a suitable constant from each function. We will use $D(\Omega)$ instead of $D(\Omega, z_0)$ if the distinguished point is irrelevant. The square of the Dirichlet norm of f is just the area of the image of

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Ω under f, counting multiplicity. It is well known that point evaluation maps on $D(\Omega)$ are bounded (see Taylor [9]). We will use $\| f \|_{D(\Omega)}^2$ to denote $\int_\Omega | f' |^2 \, dA$ even though $f \in H(\Omega)\backslash D(\Omega, z_0)$ where $H(\Omega)$ is the set of analytic functions on Ω. An Analytic function φ on Ω is called a *multiplier* of $D(\Omega)$ if $\varphi f \in D(\Omega)$ for all $f \in D(\Omega)$. Let $\varphi \in M(D(\Omega))$, the set of all multipliers of $D(\Omega)$. The linear transformation $M_\varphi : D(\Omega, z_0) \to D(\Omega, z_0)$ defined by $M_\varphi f = \varphi f$ is bounded; This follows from the Closed Graph Theorem and the boundedness of point evaluation maps. M_φ is called a multiplication operator. Giving each function in $M(D(\Omega))$ the operator norm of the corresponding multiplication operator makes $M(D(\Omega))$ into a normed space. If $\varphi \in M(D(\Omega))$, then φ is in the set of bounded analytic functions $H^\infty(\Omega)$ with $\| \varphi \|_\infty \leq \| M_\varphi \|$ (see [7], Lemma 11).

Recall that an operator T on a Hilbert space H is called *Fredholm* if the kernel of T and H/TH are both finite dimensional vector spaces. These conditions imply that T has closed range (see [2], Cor 3.2.5). Suppose T is an operator on a Hilbert space H. The *essential spectrum* of T, denoted $\sigma_e(T)$, is defined to be the set of complex numbers c such that $T - c$ is not Fredholm. $\sigma_e(T)$ is precisely the spectrum of T in the Calkin algebra $L(H)/K(H)$ where $L(H)$ denotes the set of all bounded operators on H, and $K(H)$ denotes the set of all compact operators on H (see Douglas [6]). If φ is an analytic function on Ω, then the *cluster set* of φ on $\partial\Omega$, denoted $\mathrm{cl}(\varphi; \partial\Omega)$, is the set of complex numbers c such that there exists a sequence $\{z_n\}$ in Ω such that z_n tends to $\partial\Omega$ and $f(z_n) \to c$ as $n \to \infty$.

2. Introduction

Suppose H is a normed linear space of analytic functions on Ω. Let $\{z_n\}$ be a sequence of points in Ω and let $\{c_n\}$ be a sequence of complex numbers. The interpolation problem for H is finding a function f in H such that $f(z_n) = c_n$ for all n. A sequence of points $\{z_n\}$ in Ω is called an *interpolating sequence for* H if, for every bounded sequence $\{c_n\}$, there is a function f in H such that $f(z_n) = c_n$ for all n. Carleson first solved the interpolation problem for $H^\infty(U)$, the set of bounded holomorphic functions on the open unit disk U. He proved that a necessary and sufficient condition for the given sequence $\{z_n\}$ in U to be an interpolating sequence for $H^\infty(U)$ is that the sequence $\{z_n\}$ is uniformly separated (see Carleson [3]). Later this subject was studied by many mathematicians and played a crucial role in studying closed algebras between $L^\infty(U)$ and $H^\infty(U)$, and in studying maximal ideal space of $H^\infty(U)$. Recently Chan and Shields studied a universal interpolating sequence for $D(\Omega)$; see [4]. We encountered the interpolation problem for $M(D(\Omega))$ to characterize the essential spectrum of a multiplication operator on $D(\Omega)$. We will prove that if every sequence $\{z_n\}$ in Ω converging to $\partial\Omega$ has an interpolating subsequence for $M(D(\Omega))$, then $\sigma_e(M_\varphi) = \mathrm{cl}(\varphi; \partial\Omega)$.

3. Main theorems

First we will prove that $M(D(\Omega))$ is weak-* closed in $L(D(\Omega))$, the set of all bounded operators on $D(\Omega)$. Let H be a Hilbert space. Let $S_1(H)$ denote the set of trace class operators on H, and let $F_n(H)$ denote the set of rank n operators; namely the set of operators on H whose range has dimension n. We know that $L(H) = (S_1(H))^*$ with the pairing $< S,T >= \mathrm{tr}(ST)$ where $S \in S_1(H)$, $T \in L(H)$. Here $\mathrm{tr}(ST)$ is defined to be $\sum_{j\in J} < STe_j, e_j >_H$ where $\{e_j\}_{j\in J}$ is any orthonomal basis of H (see, for example, Zhu [10], Chapter 1).

PROPOSITION 3.1. *Suppose Ω is a domain in* $\mathbf{C}$. *Then $M(D(\Omega))$ is weak-* closed in $L(D(\Omega))$.*

PROOF. Let $\{\varphi_\alpha\}_{\alpha\in A}$ be a net in $M(D(\Omega))$. Suppose $M_{\varphi_\alpha} \to T$ weak-* in $L(D(\Omega))$. We must show that there is a function φ in $H^\infty(\Omega)$ such that $T = M_\varphi$. Since $M_{\varphi_\alpha} \to T$ weak-* implies $< S, M_{\varphi_\alpha} > \to < S,T >$ for all S in $S_1(D(\Omega))$, by the pairing,

$$\mathrm{tr}(SM_{\varphi_\alpha}) \to \mathrm{tr}(ST) \tag{3.2}$$

for all S in $S_1(D(\Omega))$. Fix a point z in Ω and fix a function f in $D(\Omega)$. Let λ_z be a point evaluation map on $D(\Omega)$ at z. Define an operator $\lambda_z \otimes f : D(\Omega) \to D(\Omega)$ by

$$(\lambda_z \otimes f)(g) =< g, \lambda_z >_{D(\Omega)} f = g(z)f.$$

Let $S = \lambda_z \otimes f$. Then $S \in F_1(D(\Omega)) \subset S_1(D(\Omega))$. Let $e_1 = \dfrac{f}{\| f \|_{D(\Omega)}}$. Expand $\{e_1\}$ to an orthonomal basis $\{e_j\}_{j\in J}$ of $D(\Omega)$. Then, for all $\alpha \in A$ and $j \in J$,

$$(3.3) \quad SM_{\varphi_\alpha} e_j = S(\varphi_\alpha e_j) = \varphi_\alpha(z)e_j(z)f = \varphi_\alpha(z)e_j(z) \| f \|_{D(\Omega)} e_1.$$

Hence

$$\begin{aligned} \text{tr } (SM_{\varphi_\alpha}) &= \sum_{j\in J} < SM_{\varphi_\alpha} e_j, e_j >_{D(\Omega)} = < SM_{\varphi_\alpha} e_1, e_1 >_{D(\Omega)} \quad \text{by } (3.3) \\ &= < \varphi_\alpha(z)e_1(z)f, \frac{f}{\| f \|_{D(\Omega)}} >_{D(\Omega)} = \varphi_\alpha(z)f(z). \end{aligned}$$

$$\begin{aligned} \text{Similarly tr}(ST) &= \sum_{j\in J} < STe_j, e_j >_{D(\Omega)} = < STe_1, e_1 >_{D(\Omega)} \\ &= < (Te_1)(z)f, \frac{f}{\| f \|_{D(\Omega)}} >_{D(\Omega)} = (T(f))(z). \end{aligned}$$

Hence, by (3.2), for all z in Ω and for all f in $D(\Omega)$,

$$(3.4) \qquad \varphi_\alpha(z)f(z) \to (T(f))(z).$$

Let $g(z) = z - z_0$ where z_0 is the distinguished point in $D(\Omega)$. Then g is in $D(\Omega)$. Define a function φ on Ω by $\varphi(z) = \dfrac{T(g)(z)}{g(z)}$. Then φ is in $H(\Omega)$ since $(T(g))(z_0) = 0$, $\varphi_\alpha(z)g(z) \to (T(g))(z) = \varphi(z)g(z)$ pointwise on Ω, and so $\varphi_\alpha(z)$ converges to $\varphi(z)$ pointwise on Ω. Hence $\varphi_\alpha(z)f(z) \to \varphi(z)f(z)$ for all z in Ω and for all f in $D(\Omega)$. By (3.4), we can conclude that $T = M_\varphi$. (As a multiplier, φ is in $H^\infty(\Omega)$). Hence T is in $M(D(\Omega))$ and so $M(D(\Omega))$ is weak-* closed in $L(D(\Omega))$. Q.E.D.

Since $M(D(\Omega))$ is weak-* closed in $S_1(D(\Omega))^*$, ${}^\perp M(D(\Omega))$, the pre-anihilator of $M(D(\Omega))$, is a closed subspace of $S_1(D(\Omega))$. Hence $M(D(\Omega))$ is a dual space of a Banach space; namely
$M(D(\Omega)) = (S_1(D(\Omega))/{}^\perp M(D(\Omega)))^*$ since, for any normed linear space X and its closed subspace Y, $(X/Y)^* = Y^\perp$ and $({}^\perp Y)^\perp = Y$. We will use B to denote $S_1(D(\Omega))/{}^\perp M(D(\Omega))$ throughout this paper. We know

that the weak-* topology on $M(D(\Omega))$ as a dual space of B equals the weak-* topology on $M(D(\Omega))$ as a subspace of $L(D(\Omega)) = S_1(D(\Omega))^*$.

In what follows, ℓ^1 is the set of sequences $a = \{a_j\}$ in $\mathbf{C}$ such that $\sum_{j=1}^{\infty} | a_j | < \infty$, and ℓ^∞ is the set of bounded sequence in $\mathbf{C}$. The following Dor-Rosenthal Theorem will be needed to prove Theorem 3.5. H. P. Rosenthal proved this theorem for the real Banach space in [8] and later L. E. Dor proved the complex version of theorem that is shown here; see [5].

DOR-ROSENTHAL THEOREM. *Let B be a (complex) Banach space and let $\{g_n\}$ be a bounded sequence in B. Then $\{g_n\}$ has a subsequence $\{g_{n_j}\}$ satisfying one of the following two mutually exclusive alternatives.*

(1) *The map $\Lambda : \ell^1 \to$ {the closure of linear span of $\{g_{n_j}\}_{j=1}^{\infty}$ in B} defined by $\Lambda(a) = \sum_{j=1}^{\infty} a_j g_{n_j}$ is an isomorphism.*

(2) $\lim_{j\to\infty} < g_{n_j}, f >$ *exists for all f in B^*.*

Let Ω be a domain in $\mathbf{C}$. A sequence $\{z_j\}$ in Ω is called an *interpolating sequence for* $M(D(\Omega))$ if, for all $b \in \ell^\infty$, there exists a function f in $M(D(\Omega))$ such that $f(z_j) = b_j$ for all j in $\mathcal{N}$. Let $z \in \Omega$. Note that the point evaluation map on $M(D(\Omega))$ at z, denoted κ_z, is bounded since $| \kappa_z(\varphi) | = | \varphi(z) | \leq \| \varphi \|_\infty \leq \| M_\varphi \|$ for all $\varphi \in M(D(\Omega))$.

THEOREM 3.5. *Suppose Ω is a domain in $\mathbf{C}$. Let $\{z_n\}$ be a sequence in Ω. Then $\{z_n\}$ has a subsequence $\{z_{n_j}\}$ satisfying one of the following exclusively.*

(1) *$\{z_{n_j}\}$ is an interpolating sequence for $M(D(\Omega))$.*

(2) *$\lim_{j\to\infty} \varphi(z_{n_j})$ exists for all φ in $M(D(\Omega))$.*

PROOF. Let $\{z_n\}$ be a sequence in Ω. Since point evaluation maps on $M(D(\Omega))$ are bounded with the norms ≤ 1, for each $n \in \mathcal{N}$, there exists a function $g_n \in M(D(\Omega))^* = B^{**}$ such that $\| g_n \|_B \leq 1$ and, for all φ in $M(D(\Omega))$,

$$< \varphi, g_n > = \kappa_{z_n}(\varphi) = \varphi(z_n). \tag{3.6}$$

Apply the Dor-Rosenthal Theorem on $\{g_n\}$. Suppose $\{z_{n_j}\}$ is a subsequence of $\{z_n\}$ that satisfies (1) in the Dor-Rosenthal Theorem. Let $b \in \ell^\infty$. Define a linear map $S : \ell^1 \to \ell^\infty$ by $S(a) = \sum_{j=1}^{\infty} a_j b_j$ and define a linear map T from {the closure of linear span of $\{g_n\}_{j=i}^{\infty}$ in B} into C by $T(f) = S(\Lambda^{-1}(f))$. Since T is bounded, by the Hahn-Banach Theorem, there exists a Ψ in $B^* = M(D(\Omega))$ such that the restriction of Ψ on the closure of linear span of $\{g_{n_j}\}_{j=1}^{\infty}$ in B is T. Fix j in $\mathcal{N}$. Then $\Psi(g_{n_j}) = T(g_{n_j}) = b_j$. On the other hand $\Psi(g_{n_j}) = < g_{n_j}, \Psi > = \Psi(z_{n_j})$ by (3.6). Hence $\Psi(z_{n_j}) = b_j$ for all j and so $\{z_{n_j}\}$is an interpolating sequence for $M(D(\Omega))$. Suppose $\{z_{n_j}\}$ is a subsequence of $\{z_n\}$ that satisfies (2) in the Dor-Rosenthal Theorem. Then $\lim_{j\to\infty} < g_{n_j}, \varphi > = \lim_{j\to\infty} \varphi(z_{n_j})$ exist for all $\varphi \in M(D(\Omega))$. Q.E.D.

Suppose $\{z_n\} \subset \Omega$ is an interpolating sequence for $M(D(\Omega))$. Then a map $\Phi : M(D(\Omega)) \to \ell^\infty$ defined by $\Phi(\varphi) = \{\varphi(z_n)\}$ is onto. By the Open Mapping Theorem, there exists a constant K such that
$\|\Phi^{-1}(b)\|_{M(D(\Omega))} \le K\|b\|_{\ell^\infty}$ for all $b \in \ell^\infty$.

THEOREM 3.7. *Let $\varphi \in M(D(\Omega))$. Suppose there is a sequence $\{z_n\}$ in Ω such that $z_n \to \partial\Omega$ and $\varphi(z_n) \to 0$. If $\{z_n\}$ has an interpolating subsequence for $M(D(\Omega))$, then M_φ is not a Fredhlom operator.*

PROOF. Suppose $\{z_{n_j}\}$ is an interpolating subsequence of $\{z_n\}$. For each k in $\mathcal{N}$, let $b^k = (0, \ldots, 0, \varphi(z_{n_k}), \varphi(z_{n_{k+1}}), \ldots)$. Then, since $\varphi(z_{n_j}) \to 0$, $b^k \in \ell^\infty$ for all k in $\mathcal{N}$. Hence, for each b^k, there exists $\varphi_k \in M(D(\Omega))$ such that

$$\varphi(z_{n_j}) = \begin{cases} 0 & \text{if} \quad j < k \\ \varphi(z_{n_j}) & \text{if} \quad j > k \end{cases} \tag{3.8}$$

By the comment preceding this theorem, there is a constant K such that $\|\varphi_k\|_{M(D(\Omega))} \le K\|b^k\|_{\ell^\infty}$ for all k in $\mathcal{N}$. Since $\|b^k\|_{\ell^\infty} \to 0$ as $k \to \infty$,

$$\|\varphi_k\|_{M(D(\Omega))} \to 0 \tag{3.9}$$

as $k \to \infty$.

Note that, by (3.8),

$$(\varphi - \varphi_k)(z_{n_j}) = \begin{cases} \varphi(z_{n_j}) & \text{if} \quad j < k \\ 0 & \text{if} \quad j \geq k \end{cases}$$

Hence

$$\text{Range} M_{\varphi - \varphi_k} \subset \cap_{j=k}^{\infty} \text{Ker} \lambda_{z_{n_j}}. \tag{3.10}$$

where $\lambda_{z_{n_j}}$ is a point evaluation map on $D(\Omega)$ at z_{n_j}. Since $\{\lambda_{z_{n_j}}\}$ is a linearly independent subset of $L(D(\Omega))$, the right hand side of (3.10) has an infinite codimension and so does $\text{Range}M_{\varphi-\varphi_k}$. Therefore $M_{\varphi-\varphi_k}$ is not Fredholm. Note that

(3.11)

$$\begin{aligned} &\|M_{\varphi-\varphi_k} - M_{\varphi}\|_{L(D(\Omega))} \\ &= \sup\{\|[(\varphi - \varphi_k) - \varphi]f\|_{D(\Omega)} : f \in D(\Omega), \|f\|_{D(\Omega)} = 1\} \\ &= \sup\{\|\varphi_k f\|_{D(\Omega)} : f \in D(\Omega), \|f\|_{D(\Omega)} = 1\} \\ &\leq \|M_{\varphi_k}\|_{L(D(\Omega))} = \|\varphi_k\|_{M(D(\Omega))}. \end{aligned}$$

By (3.9), the right hand side of (3.11) tends to 0 as $k \to \infty$. Hence $M_{\varphi-\varphi_k} \to M_{\varphi}$ in $L(D(\Omega))$. Since the set of Fredholm operators is open in $L(D(\Omega))$, and $M_{\varphi-\varphi_k}$ is not a Fredholm operator for each k, M_{φ} is not a Fredholm operator. Q.E.D.

The following lemma can be proved using the mean value property of the analytic function, Hölder's inequility, the Cauchy Formula, and Uniform Boundedness Principle.

LEMMA 3.12. *Let $z \in \Omega$ and let $n \in \mathcal{N} \cup \{0\}$. Then the map $\lambda_{z,n}$: $D(\Omega, z_0) \to \mathbf{C}$ defined by $\lambda_{z,n}(f) = f^{(n)}(z)$ is a bounded linear function.*

Suppose Ω is a bounded domain in $\mathbf{C}$ such that $D(\Omega) \subset B(\Omega)$. Let $\varphi \in M(D(\Omega))$ and suppose that $0 \in \text{cl}(\varphi; \partial\Omega)$, i.e. φ is bounded away from 0 near $\partial\Omega$. We want to show that M_{φ} is Fredholm. Let $z_1, \ldots, z_n$ be the distinct zeros of φ in Ω. For $j = 1, \ldots, n$ let $m(z_j)$ be the multiplicity of the zero of φ at z_j if $z_j = z_0$, and let $m(z_j)$ be (the multiplicity of the zero of φ at z_j)+1 if $z_j = z_0$. Let E be the subspace of $D(\Omega, z_0)$ consisting of all functions f in $D(\Omega, z_0)$ such that f vanishes on $\{z_1, \ldots, z_n\}$ with multiplicity bigger than or equal to $m(z_j)$ at each z_j. Let $f \in E$. Then $\frac{f}{\varphi} \in H(\Omega)$ and $\frac{f}{\varphi}(z_0) = 0$. To see $\frac{f}{\varphi}$ is in $D(\Omega, z_0)$, observe that $\left(\frac{f}{\varphi}\right)' = \left(\frac{f'\varphi - f\varphi'}{\varphi^2}\right)$.

Since $\varphi \in M(D(\Omega))$ implies $\varphi' D(\Omega) \subset B(\Omega)$, the numerator is square integrable on Ω, and so $\frac{f}{\varphi} \in D(\Omega, z_0)$. Hence f is in the range of M_φ and so E is contained in the range of M_φ. Note that

$$E = \left\{\mathrm{Ker}\lambda_{z_j,k} : j = 1, \ldots, n \text{ and } k = 0, \ldots, m(z_j) - 1\right\}.$$

Being an intersection of the kernels of finitely many linear functionals, E has a finite codimension. Since $\mathrm{Ker} M_\varphi = \{0\}$, M_φ is Fredholm. We just prove that $\sigma_e(M_\varphi) \subset \mathrm{cl}(\varphi; \partial\Omega)$ for a bounded domain Ω in $\mathbf{C}$ such that $D(\Omega) \subset B(\Omega)$. Now suppose that every sequence in Ω converging to $\partial\Omega$ has an interpolating subsequence for $M(D(\Omega))$, then, by the above theorem, $\mathrm{cl}(\varphi; \partial\Omega) \subset \sigma_e(M_\varphi)$. Hence, to prove $\sigma_e(M_\varphi) = \mathrm{cl}(\varphi; \partial\Omega)$, it is enough to show that every sequence in Ω converging to $\partial\Omega$ has an interpolating subsequence for $M(D(\Omega))$.

Suppose Ω is a finitely connected bounded domain in C with m holes. The $m+1$ mutually disjoint simple closed curves consisting of $\partial\Omega$ will be donoted by $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$, where Γ_0 is the boundary of the unbounded component of $\mathbf{S}^2 \setminus \Omega$. Ω_0, or sometimes U_0, will be used to denote the bounded component of $\mathbf{S}^2 \setminus \Gamma_0$, and U_j will be used to denote the unbounded component of $\mathbf{S}^2 \setminus \Gamma_j$ for each $j = 1, \ldots, m$. Let A_j be connected neighborhood of Γ_j in Ω such that $A_j \cap A_k$ is an empty set if $j = k$. Using the decomposition theorem for a holomorphic function, we can show that, for each f in $D(\Omega)$, there is a function f_j in $D(\Omega_j) \cap H(U_j)$ for each $j = 1, \ldots, m$, such that $f = f_0 + f_1 + \cdots + f_m$ on Ω.

The following lemma can be proved using change-of-variables.

LEMMA 3.13. *Let Ω_1 and Ω_2 be two domains in* $\mathbf{C}$ *and let $z_0 \in \Omega_1$ and $w_0 \in \Omega_2$. Suppose Ψ is a conformal mapping from Ω_2 onto Ω_1 such* *that* $\Psi(w_0) = z_0$. *Then*

(1) *the composition map* $C_\psi : D(\Omega_1, z_0) \to D(\Omega_2, w_0)$ *defined by* $C_\Psi(f) = f \circ \Psi$ *is a unitary map,*

(2) *the composition map* $C_\Psi : M(D(\Omega_1, z_0)) \to M(D(\Omega_2, w_0))$ *defined by* $C_\Psi(\varphi) = \Psi \circ \varphi$ *is an onto isometry.*

S. Axler proved that, for every sequence $\{z_n\}$ in U converging to ∂U, there exists a multiplier φ of $D(U)$ such that $\lim_{n\to\infty} \varphi(z_n)$ does not exist; see [1]. Now we are ready to prove the following theorem.

THEOREM 3.14. *Suppose* Ω *is a finitely connected bounded domain in* $\mathbf{C}$ *such that* $D(\Omega) \subset B(\Omega)$. *If* $\{z_n\}$ *is a sequence in* Ω *converging to* $\partial\Omega$, *then there exists a multiplier* φ *of* $D(\Omega)$ *such that* $\lim_{n\to\infty} \varphi(z_n)$ *does not exist.*

PROOF. Let $\{z_n\}$ be sequence in Ω converging to $\partial\Omega$. We may assume that Γ_0 is ∂U and $\{z_n\}$ converges to Γ_0 by Lemms 3.13. Then there exists $\varphi \in M(D(\Omega))$ such that $\lim_{n\to\infty} \varphi(z_n)$ does not exist. We claim that φ is in $M(D(\Omega))$. Let $f \in D(\Omega)$. Since $(\varphi f)' = \varphi' f + \varphi f'$ and $\varphi f'$ is in $B(\Omega)$, it suffices to show that $\varphi' f$ is in $B(\Omega)$. On $\Omega \setminus A_0, \varphi'$ is bounded. Since $D(\Omega) \subset B(\Omega), \varphi' f$ is square integrable on $\Omega \setminus A_0$. By the decomposition of $f, f = f_0 + f_1 + \cdots + f_m$ where $f_j \in D(\Omega_j) \cap H(U_j)$ for each j. $\varphi' f_0$ is in $B(\Omega)$ because f_0 is in $D(U)$ which is a subset of $B(U)$. Note that f_j is bounded on A_0 because f_j is in $H(U_j)$ for each $j = 1, \ldots, m$, and $\varphi \in D(\Omega)$ implies $\varphi' \in B(\Omega)$ since Ω is bounded. Hence $\varphi' f$ is square integrable on A_0 and φ is in $M(D(\Omega))$. Q.E.D.

Suppose Ω is a finitely connected bounded domain in $\mathbf{C}$ such that $D(\Omega) \subset B(\Omega)$. By Theorem 3.5 and the above theorem, we can conclude that each sequence in Ω converging to $\partial\Omega$ has an interpolating subsequence for the $M(D(\Omega))$. Hence, by the comments following Lemma 3.12, we have the following corollary.

COROLLARY 3.15. *Suppose* Ω *is a finitely connected bounded domain in* $\mathbf{C}$ *such that* $D(\Omega) \subset B(\Omega)$. *Let* $\varphi \in M(D(\Omega))$. *Then* $\sigma_e(M_\varphi) = cl(\varphi; \partial\Omega)$.

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