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ON TRANSFORMATION OF THE DENSITY OF THE THINNEST COVERING AND ITS APPLICATIONS

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ABSTRACT. The expression of the density of the thinnest covering of $\mathbb{R}^n$ shall be transformed into a more 'available' form, and some applications of the transformed expression shall be introduced.

1. Introduction

Throughout this note, let $\mathbb{R}^n$ be the n-dimensional euclidean space and $K \subset \mathbb{R}^n$ a bounded set, with a positive Lebesgue measure, which contains the origin of the coordinate system. By $K + a$ we denote the translation of K by a, where a is an arbitrary point in $\mathbb{R}^n$. Set mK to be the set of all mx with $x \in K$, where m is an arbitrary real number. We use $\mathcal{U}(K)$ to denote the set of all countably infinite coverings of $\mathbb{R}^n$ of the form $\{K + a_i\}_i$, where the a_i are the points in $\mathbb{R}^n$. By W we denote a half-open cube with its sides parallel to the coordinate axes, e.g., the set defined by

$$-s \le x_1 < s, \quad -s \le x_2 < s, \quad \cdots, \quad -s \le x_n < s,$$

where s is a positive real number. We denote by μ the Lebesgue measure on $\mathbb{R}^n$.

The *density of the thinnest* (or *the most economical*) *covering of the whole space by translates* of K is defined by

$$\vartheta(K) = \inf_{\{K+a_i\}_i \in \mathcal{U}(K)} \liminf_{s\to\infty} \frac{1}{\mu(W)} \sum_{K+a_i \subset W} \mu(K + a_i). \tag{1}$$

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But it follows from theorem 1.10 in [7] that

$$\vartheta(K) = \inf_{\{K+a_i\}_i \in \mathcal{U}(K)} \limsup_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+a_i)\cap W \neq \emptyset} \mu(K+a_i).$$

Comparing the last expression and (1) we can confine $\mathcal{U}(K)$ to the set of all countably infinite coverings of $\mathbb{R}^n$ of the form $\{K+a_i\}_i$ satisfying

$$\lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+a_i)\cap W \neq \emptyset} \mu(K+a_i) < \infty, \tag{2}$$

and easily verify that

$$\begin{aligned} \vartheta(K) &= \inf_{\{K+a_i\}_i \in \mathcal{U}(K)} \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+a_i)\cap W \neq \emptyset} \mu(K+a_i) \\ &= \inf_{\{K+a_i\}_i \in \mathcal{U}(K)} \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{K+a_i \subset W} \mu(K+a_i). \end{aligned} \tag{3}$$

The set $\mathcal{U}(K)$ is not empty, since every lattice covering of the form $\{K+a_i\}_i$ belongs to it. The covering of the whole space $\mathbb{R}^n$ is a very interesting problem which is intensively investigated (see [1]–[8]).

The main result of this note is to convert the expression (1) into

$$\vartheta(K) = \lim_{m\to 0+} \inf_{\{mK+a_i\}_i \in \mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G \neq \emptyset} \mu(mK+a_i),$$

where G is an open cube with its centre at the origin of the coordinate system. Roughly speaking, the places of the 'inf' and 'lim' in the definition (1) are interchanged, and we know that it is not trivial to interchange their orders.

In section 4 we apply this result to investigate the properties of compact null sets (they are surely nowhere dense).

2. Preliminaries

LEMMA. *1. Let $m > 0$ be an arbitrary real number. Then the cardinal numbers of $\mathcal{U}(K)$ and $\mathcal{U}(mK)$ are equivalent.*

PROOF. It is enough to show

$$\mathcal{U}(mK) = \{\{mK + ma_i\}_i \mid \{K + a_i\}_i \in \mathcal{U}(K)\}.$$

Assume that $\{K + a_i\}_i \in \mathcal{U}(K)$ and $x \in \mathbb{R}^n$. As $\{K + a_i\}_i$ is a covering of $\mathbb{R}^n$, there is an $i(x) \in \mathbb{N}$ with $x/m \in K + a_{i(x)}$, and so there is a $z \in K$ with $x/m = z + a_{i(x)}$, i.e., there is an $mz \in mK$ with $x = mz + ma_{i(x)}$. Hence $x \in mK + ma_{i(x)}$. As x is an arbitrary point in $\mathbb{R}^n$, $\{mK + ma_i\}_i$ is a covering of $\mathbb{R}^n$. As $(mK + ma_i) \cap W \neq \emptyset$ if and only if $(K + a_i) \cap (1/m)W \neq \emptyset$, we have

$$\begin{aligned}
&\lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(mK+ma_i)\cap W\neq\emptyset} \mu(mK + ma_i)\\
&= \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+a_i)\cap\frac{1}{m}W\neq\emptyset} \mu(mK + ma_i)\\
&= \lim_{s\to\infty} \frac{1}{\mu(\frac{1}{m}W)} \sum_{(K+a_i)\cap\frac{1}{m}W\neq\emptyset} \mu(K + a_i)\\
&= \lim_{s/m\to\infty} \frac{1}{\mu(W)} \sum_{(K+a_i)\cap W\neq\emptyset} \mu(K + a_i)\\
&<\infty.
\end{aligned}$$

Thus it holds that

$$\begin{aligned}
&\{mK + ma_i\}_i \in \mathcal{U}(mK) \quad \text{and}\\
&\{\{mK + ma_i\}_i \mid \{K + a_i\}_i \in \mathcal{U}(K)\} \subset \mathcal{U}(mK).
\end{aligned}$$

The opposite inclusion can be analogously proved.

Let G be the open cube with its sides parallel to the coordinate axes and its centre at the origin of the coordinate system, i.e., the set defined by

$$-t < x_1 < t, -t < x_2 < t, \cdots, -t < x_n < t,$$

where t is a fixed positive real number.

LEMMA 2. *We obtain*

$$\limsup_{m\to 0+} \inf_{\{mK+a_i\}_i\in\mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i) \leq \vartheta(K).$$

PROOF. For a fixed $m>0$, suppose that $\{K+a_i/m\}_i\in\mathcal{U}(K)$. Then we have

$$\begin{aligned}
&\inf_{\{mK+c_i\}_i\in\mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+c_i)\cap G\neq\emptyset} \mu(mK+c_i) \leq \\
&\leq \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i) \\
&\leq \sup_{s\geq\frac{t}{m}} \frac{1}{\mu(W)} \sum_{(K+\frac{1}{m}a_i)\cap W\neq\emptyset} \mu\left(K+\frac{1}{m}a_i\right),
\end{aligned}$$

since $(K+a_i/m)\cap W\neq\emptyset$ if and only if $(mK+a_i)\cap mW\neq\emptyset$ and $G\subset mW$ if $s\geq t/m$. By the last inequality and (2) we obtain

$$\begin{aligned}
&\limsup_{m\to 0+} \inf_{\{mK+c_i\}_i\in\mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+c_i)\cap G\neq\emptyset} \mu(mK+c_i) \leq \\
&\leq \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+\frac{1}{m}a_i)\cap W\neq\emptyset} \mu\left(K+\frac{1}{m}a_i\right) \quad \text{for every} \\
&\{K+a_i/m\}_i\in\mathcal{U}(K).
\end{aligned}$$

Combining the last inequality and (3) we have

$$\begin{aligned}
&\limsup_{m\to 0+} \inf_{\{mK+c_i\}_i\in\mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+c_i)\cap G\neq\emptyset} \mu(mK+c_i) \leq \\
&\leq \inf_{\{K+a_i\}_i\in\mathcal{U}(K)} \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+a_i)\cap W\neq\emptyset} \mu(K+a_i) = \vartheta(K),
\end{aligned}$$

as required.

LEMMA 3. *Let W° be the interior of W. For any $\{K+a_i\}_i \in \mathcal{U}(K)$ we have*

$$\lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+a_i)\cap W^\circ \neq \emptyset} \mu(K+a_i)$$
$$= \lim_{m\to 0+} \frac{1}{\mu(G)} \sum_{(mK+ma_i)\cap G\neq\emptyset} \mu(mK+ma_i).$$

PROOF. Let s, t be any positive real numbers. Then $(K+a_i)\cap W^\circ \neq \emptyset$ if and only if $((t/s)K+(t/s)a_i)\cap G \neq \emptyset$. Set $m = t/s$. Then

$$\lim_{s\to\infty} \frac{1}{(2s)^n} \sum_{(K+a_i)\cap W^\circ\neq\emptyset} \mu(K+a_i)$$
$$= \lim_{m\to 0+} \frac{1}{(2t/m)^n} \sum_{(\frac{t}{s}K+\frac{t}{s}a_i)\cap G\neq\emptyset} \left(\frac{s}{t}\right)^n \mu\left(\frac{t}{s}K+\frac{t}{s}a_i\right)$$
$$= \lim_{m\to 0+} \frac{1}{(2t)^n} \sum_{(mK+ma_i)\cap G\neq\emptyset} \mu(mK+ma_i),$$

as required.

3. The main result

According to (3) the density of the thinnest covering of the whole space by translates of K is estimated by calculating the ratio of the sum of the Lebesgue measures of translates of K, which cover the half-open cube W, to the Lebesgue measure of W as the cube W expands to infinity and then taking the infimum over all coverings of $\mathbb{R}^n$ by translates of K.

But, by the next theorem, we can also do it by calculating the ratio of the sum of the Lebesgue measures of translates of mK, which cover the open cube G, to the Lebesgue measure of G and taking the infimum over all coverings of G by translates of mK and then letting m decrease to 0.

THEOREM 4. *Let $K \subset \mathbb{R}^n$ be a bounded set with a positive Lebesgue measure, which includes the origin of the coordinate system. Then*

$$\vartheta(K) = \lim_{m\to 0+} \inf_{\{mK+a_i\}_i \in \mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i).$$

PROOF. Let $\varepsilon > 0$ be an arbitrary real number and $\{K+e_i\}_i \in \mathcal{U}(K)$ with

$$\lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+e_i)\cap W^\circ\neq\emptyset} \mu(K+e_i)$$
$$\leq \inf_{\{K+a_i\}_i \in \mathcal{U}(K)} \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+a_i)\cap W^\circ\neq\emptyset} \mu(K+a_i) + \varepsilon.$$

It follows from lemma 3 that there is a sufficiently small $m_0 > 0$ such that

$$\Bigg|\frac{1}{\mu(G)} \sum_{(mK+me_i)\cap G\neq\emptyset} \mu(mK+me_i) - \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+e_i)\cap W^\circ\neq\emptyset} \mu(K+e_i)\Bigg| < \varepsilon \tag{4}$$

for all $0 < m \leq m_0$. Assume that $\{K+b_i\}_i \in \mathcal{U}(K)$ does not satisfy the inequality (4). Then there is an $m, 0 < m \leq m_0$, satisfying

$$\Bigg|\frac{1}{\mu(G)} \sum_{(mK+mb_i)\cap G\neq\emptyset} \mu(mK+mb_i) - \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+b_i)\cap W^\circ\neq\emptyset} \mu(K+b_i)\Bigg| \geq \varepsilon.$$

On account of (2) and lemma 3, there exists a positive number M satisfying

$$\lim_{m'\to 0} \frac{1}{\mu(G)} \sum_{(m'K+m'b_i)\cap G\neq\emptyset} \mu(m'K+m'b_i) < M.$$

As m (or m_0) is sufficiently small, we may assume that

$$\frac{1}{\mu(G)} \sum_{(mK+mb_i)\cap G\neq\emptyset} \mu(mK+mb_i) < 2M.$$

Let G' be an open cube defined by

$$-t+\delta < x_i < t-\delta (i=1,...,n)$$

with

$$\delta \leq t\left(1-\left(1+\frac{\varepsilon}{4M}\right)^{-1/n}\right).$$

We can choose such a small m (or m_0) that $md(K)/\delta$ becomes small enough, where $d(K)$ denotes the *diameter* of K.

Without loss of generality, suppose that there is a $p \in \mathbb{N}$ such that $(mK+mb_i)\cap G' \neq \emptyset$ for $i \in \{1,...,p\}$ and $(mK+mb_i)\cap G' = \emptyset$ otherwise. Let $\{d_j\}_{j\in\mathbb{N}}$ be an enumeration of the lattice of all points that have coordinates that are integral multiples of $2t$. (Recall that the side length of G is $2t$). Let $\{mK+g_i\}_i$ be one of the thinnest coverings of $\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty}(G'+d_j)$, and let $\{mK+c_i\}_i$ be an enumeration of $\{mK+mb_i+d_j\}_{i=1,...,p;j\in\mathbb{N}} \cup \{mK+g_i\}_i$. Then, in view of the definition of the thinnest covering and the homogeneity of the space $\mathbb{R}^n$,

$$\begin{aligned}&\frac{1}{\mu(G\setminus G')} \sum_{(mK+c_i)\cap(G\setminus G')\neq\emptyset} \mu(mK+c_i)\\ \leq& 2\frac{1}{\mu(G')} \sum_{(mK+mb_i)\cap G'\neq\emptyset} \mu(mK+mb_i).\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{\mu(G)} \sum_{(mK+c_i)\cap G\neq\emptyset} \mu(mK+c_i)\\ \leq&\frac{1}{\mu(G)} \sum_{(mK+mb_i)\cap G'\neq\emptyset} \mu(mK+mb_i)+\\ &+\frac{1}{\mu(G)} \sum_{(mK+c_i)\cap(G\setminus G')\neq\emptyset} \mu(mK+c_i)\\ \leq&\frac{1}{\mu(G)} \sum_{(mK+mb_i)\cap G\neq\emptyset} \mu(mK+mb_i)+\varepsilon. \qquad (5)\end{aligned}$$

It is easy to show that $\{K+(1/m)c_i\}_i \in \mathcal{U}(K)$. Hence, owing to lemma 3,

$$\Big|\frac{1}{\mu(G)} \sum_{(mK+c_i)\cap G\neq\emptyset} \mu(mK+c_i) - \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+\frac{1}{m}c_i)\cap W^\circ\neq\emptyset} \mu\left(K+\frac{1}{m}c_i\right)\Big| < \varepsilon,$$

since m is sufficiently small. For all $0 < m \le m_0$, set $\mathcal{U}'(mK)$ to be the set of all $\{mK+a_i\}_i \in \mathcal{U}(mK)$ satisfying

$$\Big|\frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i) - \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+\frac{1}{m}a_i)\cap W^\circ\neq\emptyset} \mu\left(K+\frac{1}{m}a_i\right)\Big| < \varepsilon. \tag{6}$$

Consequently, for every $\{mK+mb_i\}_i \in \mathcal{U}(mK)\setminus\mathcal{U}'(mK)$ there exists a $\{mK+c_i\}_i \in \mathcal{U}'(mK)$ satisfying (5). Accordingly,

$$\inf_{\{mK+a_i\}_i\in\mathcal{U}'(mK)} \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i) \le \tag{7}$$

$$\le \inf_{\{mK+a_i\}_i\in\mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i) + \varepsilon.$$

Choose a $\{mK+f_i\}_i \in \mathcal{U}'(mK)$ such that

$$\frac{1}{\mu(G)} \sum_{(mK+f_i)\cap G\neq\emptyset} \mu(mK+f_i) \le \tag{8}$$

$$\le \inf_{\{mK+a_i\}_i\in\mathcal{U}'(mK)} \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i) + \varepsilon.$$

It follows then from (7) and (8) that

$$\frac{1}{\mu(G)} \sum_{(mK+f_i)\cap G\neq\emptyset} \mu(mK+f_i) \le$$

$$\le \inf_{\{mK+a_i\}_i\in\mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i) + 2\varepsilon. \tag{9}$$

As $\{mK+f_i\}_i \in \mathcal{U}'(mK)$, we obtain by (6) and (9)

$$\lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+\frac{1}{m}f_i)\cap W^\circ\neq\emptyset} \mu(K+\frac{1}{m}f_i) <$$

$$< \inf_{\{mK+a_i\}_i\in\mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i)+3\varepsilon,$$

and moreover

$$\inf_{\{K+a_i\}_i\in\mathcal{U}(K)} \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+a_i)\cap W^\circ\neq\emptyset} \mu(K+a_i) <$$

(10)

$$< \inf_{\{mK+a_i\}_i\in\mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i)+3\varepsilon.$$

Since the inequality (10) holds for all $0<m\leq m_0$, we have

$$\inf_{\{K+a_i\}_i\in\mathcal{U}(K)} \lim_{s\to\infty} \frac{1}{\mu(W)} \sum_{(K+a_i)\cap W^\circ\neq\emptyset} \mu(K+a_i) \leq$$

$$\leq \liminf_{m\to 0+} \inf_{\{mK+a_i\}_i\in\mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i)+3\varepsilon.$$

As $\varepsilon>0$ may be sufficiently small, we obtain by (3)

$$\vartheta(K) \leq \liminf_{m\to 0+} \inf_{\{mK+a_i\}_i\in\mathcal{U}(mK)} \frac{1}{\mu(G)} \sum_{(mK+a_i)\cap G\neq\emptyset} \mu(mK+a_i).$$

The opposite inequality follows from lemma 2.

COROLLARY 5. *Let G' be a half-open cube satisfying $G\subset G'\subset \overline{G}$. Then*

$$\vartheta(K) = \lim_{m\to 0+} \inf_{\{mK+a_i\}_i\in\mathcal{U}(mK)} \frac{1}{\mu(G')} \sum_{mK+a_i\subset G'} \mu(mK+a_i).$$

4. Applications

Let $B_r(x)$ denote the closed ball of radius r with centre x. For any subset E of $\mathbb{R}^n$ we define a set function as

$$\nu(E) = \liminf_{\delta \to 0} \Big\{ \sum_i \mu\left(B_\delta(x_i)\right) \mid E \subset \bigcup_i B_\delta(x_i) \Big\},$$

where we merely consider coverings of E by the collection of the closed balls of equal radius. Then ν is no measure on $\mathbb{R}^n$, but it has remarkable covering property which we shall see below.

In view of theorem 4 and the definition of ν, it is not difficult to show

$$\nu(G) = \vartheta\left(B_1(0)\right) \mu(G).$$

We can easily generalize this result:

$$\nu(I) = \vartheta\left(B_1(0)\right) \mu(I)$$

for each n-dimensional interval I with $\mu(I) > 0$. But we know also that $\mu(I \cap Q^n) = 0$ but $\nu(I \cap Q^n) = \nu(I) > 0$ for all n-dimensional intervals I with $\mu(I) > 0$, where Q^n denotes the set of all n-tuple rational numbers.

The following theorem states that the value of ν vanishes for every compact null set.

THEOREM 6. *Let N be a compact null set i.e., a compact set of Lebesgue measure 0. Then*

$$\nu(N) = 0.$$

PROOF. Let $\varepsilon > 0$ be arbitrarily small and $\{I_i\}_i$ an open covering of N consisting of cubes satisfying

$$\sum_i \mu(I_i) < \varepsilon. \tag{11}$$

As N is compact, we can choose a finite subcollection that covers N. Without loss of generality, suppose that this subcollection consists of $I_1, \ldots, I_q$. Let

$$\gamma = \min\{d(I_i) \mid i = 1, \ldots, q\},$$

where $d(I_i)$ denotes the diameter of I_i. We can choose such a small $\delta > 0$ that δ/γ becomes sufficiently small. Let I'_i $(i = 1, ..., q)$ be the cube concentric with the I_i, whose side length is three times as long as that of the I_i. On account of theorem 4, we obtain

$$\begin{aligned}
&\inf\Big\{\sum_j \mu\,(B_\delta(x_j)) \mid I'_i \subset \bigcup_j B_\delta(x_j)\Big\} \\
=&\mu(I'_i)\frac{1}{\mu(I'_i)}\inf\{\cdots\} \\
\leq&\mu(I'_i)2\vartheta\,(B_1(0)),
\end{aligned}$$

as δ/γ is sufficiently small. By the last inequality, the definition of I'_i and (11), we have

$$\begin{aligned}
&\inf\Big\{\sum_j \mu\,(B_\delta(x_j)) \mid N \subset \bigcup_j B_\delta(x_j)\Big\} \leq \\
\leq&\sum_{i=1}^{q}\inf\Big\{\sum_j \mu\,(B_\delta(x_j)) \mid I'_i \subset \bigcup_j B_\delta(x_j)\Big\} \\
\leq&2\vartheta\,(B_1(0))\sum_{i=1}^{q}\mu(I'_i) \\
\leq&2\vartheta\,(B_1(0))\,3^n\varepsilon.
\end{aligned}$$

Consequently, due to the last inequality

$$\begin{aligned}
\nu(N) = &\lim_{\delta\to 0}\inf\Big\{\sum_j \mu\,(B_\delta(x_j)) \mid N \subset \bigcup_j B_\delta(x_j)\Big\} \\
\leq&2\vartheta\,(B_1(0))\,3^n\varepsilon.
\end{aligned}$$

Finally, since ε may be arbitrarily small and $\vartheta\,(B_1(0)) < \infty$,

$$\nu(N) = 0,$$

as required.

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