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ISOMETRIES OF $\mathcal{A}_{2n}^{(2)}$

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ABSTRACT. In this paper, we introduce the generalization $\mathcal{A}_{2n}^{(2)}$ of tridiagonal algebras $\mathcal{A}_{2n}$ and investigate the isometries of such algebras.

1. Introduction

The study of self-adjoint operator algebras on Hilbert space is well established. By contrast, non-self-adjoint algebras, particularly reflexive algebras, are only begun to be studied by W. B. Arveson [1] in 1974. The sequence $\mathcal{A}_2, \mathcal{A}_4, \cdots, \mathcal{A}_\infty$ of tridiagonal algebras, discovered by F. Gilfeather and D. Larson [2], is one of the most important classes of non-self-adjoint reflexive CSL-algebras.

Let $\mathcal{H}$ be a $2n$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$. Then A is in $\mathcal{A}_{2n}$ if and only if A has the form

$$\begin{pmatrix} * & * & & & & & * \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & \cdot & & \\ & & & & & \cdot & \\ & & & & & \cdot & * \\ & & & & & & * \end{pmatrix},$$

where all non-starred entries are zero and with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$. If we write the given basis in the order $\{e_1, e_3, \cdots,$

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$e_{2n-1}, e_2, e_4, \cdots, e_{2n}\}$, then the above matrix has the form

$$\begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix},$$

where D_1 and D_2 are $n \times n$ diagonal matrices and S is an $n \times n$ matrix with $*$ in the (i,i)-, $(j+1,j)$-, and $(1,n)$-components and 0 elsewhere $(1 \le i \le n, 1 \le j \le n-1)$. The algebra of all such matrices is unitarily equivalent to the tridiagonal algebras $\mathcal{A}_{2n}$. As a generalization, we consider the matrix S which has two $*$ in each row and each column. Then the collection of all matrices of the form $\begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ gives a new algebra which we could call $\mathcal{A}_{2n}^{(2)}$.

In this paper the following are proved:

(1) The algebra $\mathcal{A}_{2n}^{(2)}$ is unitarily equivalent to a direct sum of the tridagonal algebras of small sizer:

$$\mathcal{A}_{2n}^{(2)} \simeq \oplus_{i=1}^{k} \mathcal{A}_{2n_i} (n_i \ge 2, n_1 + n_2 + \cdots + n_k = n)$$

(2) A map $\varphi : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \to \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ is an isometry if and only if there exist isometries $\varphi_i : \mathcal{A}_{2n_i} \to \mathcal{A}_{2n_i}$ for all $i = 1, 2, \cdots, k$ such that $\varphi = \oplus_{i=1}^{k} \varphi_i$.

2. Preliminaries and Examples

Let $\mathcal{H}$ be a complex Hilbert space. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, then $Alg\mathcal{L}$ is the algebra of all bounded operators acting on $\mathcal{H}$ that leave invariant every orthogonal projections in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$, containing 0 and I. Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra consisting of all bounded operators acting on $\mathcal{H}$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$. A lattice $\mathcal{L}$ is commutative if each pair of projections in $\mathcal{L}$ commutes. If $\mathcal{L}$ is a commutative subspace lattice, then $Alg\mathcal{L}$ is called a CSL-algebra. By an isometry of an operator algebra $\mathcal{A}$ we mean a linear map $\varphi : \mathcal{A} \to \mathcal{A}$ such that $\|\varphi(A)\| = \|A\|$ for every A in $\mathcal{A}$. Let $\mathcal{A}$ and $\mathcal{B}$ be subalgebras of $\mathcal{B}(\mathcal{H})$, We say that $\mathcal{A}$ and $\mathcal{B}$ are unitarily equivalent if there exists a unitary operator U such that $U\mathcal{A}U^* = \mathcal{B}$ ($U\mathcal{A}U^* = \{UAU^* : A \in \mathcal{A}\}$). In this case, we write $\mathcal{A} \simeq \mathcal{B}$. Let $\mathcal{A}_i$ be a subalgebra of $\mathcal{B}(\mathcal{H})$ for all $i = 1, 2, \cdots n$. Then $\oplus_{i=1}^n \mathcal{A}_i$ is the algebra such that an operator A is in $\oplus_{i=1}^n \mathcal{A}_i$ if and only if $A = \oplus_{i=1}^n A_i$, where A_i is in $\mathcal{A}_i$ for all $i = 1, 2, \cdots, n$. If $\varphi_i : \mathcal{A}_i \to \mathcal{A}_i$ is an isometry for all $i = 1, 2, \cdots, n$, then $\oplus_{i=1}^n \varphi_i$ means the map from $\oplus_{i=1}^n \mathcal{A}_i$ into $\oplus_{i=1}^n \mathcal{A}_i$ defined by $(\oplus_{i=1}^n \varphi_i)(\oplus_{i=1}^n A_i) = \oplus_{i=1}^n \varphi_i(A_i)$. Let i and j be two nonzero natural numbers. Then E_{ij} is the matrix whose (i,j)-component is 1 and all other entries are 0. An $n \times n$ matrix D_n is said to be the backward identity matrix if the $(i, n-i+1)$-component is 1 for all $i = 1, 2, \cdots n$ and all other entries are 0. We denote the $n \times n$ identity matrix by I_n. If $x_1, x_2, \cdots, x_m \in \mathcal{H}$, then $[x_1, x_2, \cdots, x_m]$ means the closed subspace of $\mathcal{H}$ generated by the vectors $x_1, x_2, \cdots, x_m$.

EXAMPLE 2.1. Let $S_0 = \sum_{i=1}^n E_{ii} + \sum_{j=1}^{n-1} E_{j+1,j} + E_{1,n}$ be a $n \times n$ matrix and let $\begin{pmatrix} I_n & S_0 \\ \mathbf{0} & I_n \end{pmatrix}$ be in $\mathcal{A}_{2n}^{(2)}$. If we put

$$U = \sum_{k=1}^{n} (E_{2k-1,k} + E_{2k,n+k})$$

Then $U\mathcal{A}_{2n}^{(2)}U^* = \mathcal{A}_{2n}$.

EXAMPLE 2.2. Let

$$S_0 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} I_5 & S_0 \\ \mathbf{0} & I_5 \end{pmatrix} \in \mathcal{A}_{10}^{(2)}.$$

Let U be the permutation matrix induced by the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 7 & 2 & 6 & 5 & 8 & 3 & 10 & 4 & 9 \end{pmatrix}.$$

Then $U\mathcal{A}_{10}^{(2)}U^* = \mathcal{A}_{10}$.

EXAMPLE 2.3. Let

$$S_0 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} I_6 & S_0 \\ \mathbf{0} & I_6 \end{pmatrix} \in \mathcal{A}_{12}^{(2)}.$$

Let U be the permutation matrix induced by the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 7 & 3 & 9 & 2 & 8 & 4 & 11 & 5 & 12 & 6 & 10 \end{pmatrix}.$$

Then $U\mathcal{A}_{12(2)}U^* = \mathcal{A}_4 \oplus \mathcal{A}_8$.

Let $J : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be a map defined by $J(x_1, x_2, \cdots, x_{2n})^t = (\overline{x}_{2n}, \overline{x}_{2n-1}, \cdots, \overline{x}_1)^t$ for every $(x_1, x_2, \cdots, x_{2n})^t$ in $\mathbb{C}^{2n}$. Then J is a conjugation.

LEMMA 2.4. *Let A be a $2n \times 2n$ matrix. Then $(JAJ)^* = JA^*J$ and $JA^*J = D_{2n}A^tD_{2n}$, where D_{2n} is the $2n \times 2n$ backward identity matrix and A^t is the transposed matrix of A.*

The isometric maps of $\mathcal{A}_{2n}$ are characterized in [3]. From these results and Lemma 2.4, we have the following theorem.

THEOREM 2.5. *Let $\varphi : \mathcal{A}_{2n} \to \mathcal{A}_{2n}$ be an isometry such that $\varphi(I) = I$. Then there exists a unitary operator V such that $\varphi(A) = VAV^*$ or $\varphi(A) = VA^tV^*$ for all A in $\mathcal{A}_{2n}$.*

3. Direct Sum of Tridiagonal Algebras

DEFINITION 3.1. *An $n \times n$ matrix $T = [t_{ij}]$ has a cyclic chain if there is a finite sequence $t_{i_1 j_1}, t_{i_2 j_1}, t_{i_2, j_2}, t_{i_3 j_2}, t_{i_3 j_3} \cdots, t_{i_n j_n}, t_{i_1, j_n}$ of elements in T such that $t_{i_1 j_1} t_{i_2 j_1} t_{i_2, j_2} t_{i_3 j_2} t_{i_3 j_3} \cdots t_{i_n j_n} t_{i_1, j_n} \neq 0$.*

THEOREM 3.2. *If an $n \times n$ matrix T_0 has a cyclic chain and $A = \begin{pmatrix} I_n & T_0 \\ \mathbf{0} & I_n \end{pmatrix}$ is in $\mathcal{A}_{2n}^{(2)}$, then $\mathcal{A}_{2n}^{(2)}$ is unitarily equivalent to $\mathcal{A}_{2n}$.*

PROOF. Suppose T has a cyclic chain $t_{i_1 j_1}, t_{i_2 j_1}, \cdots, t_{i_n j_n}, t_{i_1, j_n}$. Let U be a $(0,1)$-matrix whose (k, i_k)-component is 1 and let V be a $(0,1)$-matrix whose (j_k, k)-component is 1. Then

$$UTV = \begin{pmatrix} t_{i_1,j_1} & & & & t_{i_1,j_n} \\ t_{i_2,j_1} & t_{i_2,j_2} & & & \\ & t_{i_3,j_2} & t_{i_3,j_3} & & \\ & & \ddots & \ddots & \\ & & & & t_{i_n,j_n} \end{pmatrix}$$

Let $W = \begin{pmatrix} U & 0 \\ 0 & V^* \end{pmatrix}$. Then $W\mathcal{A}_{2n}^{(2)}W^* \simeq \mathcal{A}_{2n}$

THEOREM 3.3. *An algebra $\mathcal{A}_{2n}^{(2)}$ is unitarily equivalent to a direct sum of the tridagonal algebras of small sizer:*

$$\mathcal{A}_{2n}^{(2)} \simeq \mathcal{A}_{2n_1} \oplus \mathcal{A}_{2n_2} \oplus \cdots \oplus \mathcal{A}_{2n_k} (n_i \geq 2, n_1 + n_2 + \cdots + n_k = n).$$

PROOF. We define a bijective correspondence between the set of all algebras $\mathcal{A}_{2n}^{(2)}$ and a set of certain $(0,1)$-matrices as follows: Given an algebra $\mathcal{A}_{2n}^{(2)}$ define a matrix $A = [a_{ij}]_{1 \leq i,j \leq 2n}$ by setting $a_{ij} = 1$ if $E_{ij} \in \mathcal{A}_{2n}^{(2)}$ and $a_{ij} = 0$ otherwise. Then $A = \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}$, where S is an $n \times n$ matrix with two 1 in each row and each column, and 0 elsewhere as entries. Conversely, such a matrix gives rise to one $\mathcal{A}_{2n}^{(2)}$.

A permutation $\sigma = \begin{pmatrix} 1 & 2 & . & . & . & 2n \\ \sigma(1) & \sigma(2) & . & . & . & \sigma(2n) \end{pmatrix}$ induces a $2n \times 2n$ unitary matrix U such that $UAU^* = [a_{\sigma(i),\sigma(j)}]$ for all $A = [a_{ij}]$ in $\mathcal{B}(\mathcal{H})$. So, in view of the foregoing, it suffices to show that by some permutation σ (i.e., by transforming simultaneously the i-th row to $\sigma(i)$-th row and the j-th column to $\sigma(j)$-column), A turns to a matrix of the form $A' = \begin{pmatrix} I_n & S' \\ 0 & I_n \end{pmatrix}$, where

$$S' = \oplus_{i=1}^{k} S_i \text{ and } S_i = \sum_{p=1}^{n_i} E_{pp} + \sum_{q=1}^{n_i - 1} E_{q+1,q} + E_{1,n_i}$$

for all $i = 1, 2, \cdots, k$. Suppose $1 \leq i < j \leq n$ or $n+1 \leq i < j \leq 2n$. Then application of the transposition (i, j) to A does not change I_n as a result. The effect of the application on S is, in the former case, an exchange of the i-th and j-th rows, and in the latter case, an exchange of the i-th and j-th columns. Hence it suffices to show that by repeating exchanges of two rows and of two columns we make any matrix S into a matrix S'. But this can be checked easily.

If $\mathcal{A}'^{(2)}_{2n}$ is the algebra corresponding to the resulting matrix A' by the transformation of A and if we put

$$F_1 = \sum_{i=1}^{n_1} (E_{ii} + E_{n+i,n+i}),$$

$$F_2 = \sum_{i=n_1+1}^{n_1+n_2} (E_{ii} + E_{n+i,n+i}),$$

$$\cdots\cdots$$

$$F_k = \sum_{i=n_1+\cdots+n_{k-1}+1}^{n_1+\cdots+n_k} (E_{ii} + E_{n+i,n+i}).$$

where $n_1 + n_2 + \cdots + n_k = n$, then $F_i\mathcal{A}'^{(2)}_{2n}$ is unitary equivalent to $\mathcal{A}_{2n_i} (i = 1, 2, \cdots, k)$ and hence $\mathcal{A}'^{(2)}_{2n}$ is unitary equivalent to $\mathcal{A}_{2n_1} \oplus \mathcal{A}_{2n_2} \oplus \cdots \oplus \mathcal{A}_{2n_k}$.

Let $\mathcal{H}$ be a $2n$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$ and let $n_i \geq 2 (i = 1, 2, \cdots, k)$ and $n_1 + n_2 + \cdots + n_k = n$. Let $\mathcal{L}_1$ be the subspace lattice of orthogonal projections generated by

$$\{[e_1], [e_3], \cdots, [e_{2n_1-1}], [e_1, e_2, e_3], [e_3, e_4, e_5],$$
$$\cdots, [e_{2n_1-3}, e_{2n_1-2}, e_{2n_1-1}], [e_1, e_{2n_1-1}, e_{2n_1}]\},$$

and let $\mathcal{L}_{j+1}$ be the subspace lattice of orthogonal projections generated by

$$\begin{aligned}
&\{[e_{2n_1+\cdots+2n_j+1}], [e_{2n_1+\cdots+2n_j+3}], \cdots, [e_{2n_1+\cdots+2n_{j+1}-1}],\\
&[e_{2n_1+\cdots+2n_j+1}, e_{2n_1+\cdots+2n_j+2}, e_{2n_1+\cdots+2n_j+3}],\\
&[e_{2n_1+\cdots+2n_j+3}, e_{2n_1+\cdots+2n_j+4}, e_{2n_1+\cdots+2n_j+5}], \cdots,\\
&[e_{2n_1+\cdots+2n_{j+1}-3}, e_{2n_1+\cdots+2n_{j+1}-2}, e_{2n_1+\cdots+2n_{j+1}-1}],\\
&[e_{2n_1+\cdots+2n_j+1}, e_{2n_1+\cdots+2n_{j+1}-1}, e_{2n_1+\cdots+2n_{j+1}},]\}.
\end{aligned}$$

for all $j = 1, 2, \cdots, k-1$. Let $\mathcal{L} = \bigvee_{i=1}^{k} \mathcal{L}_i$. Then $\oplus_{i=1}^{k} \mathcal{A}_{2n_i} = Alg\mathcal{L}$

THEOREM 3.4. *An algebra $\mathcal{A}_{2n}^{(2)}$ is a non-self-adjoint reflexive CSL-algebras.*

4. Isometries of $\mathcal{A}_{2n}^{(2)}$

Let $\phi : \mathcal{A}_{2n}^{(2)} \rightarrow \mathcal{A}_{2n}^{(2)}$ be an isometry. Since the algebras $\mathcal{A}_{2n}^{(2)}$ and $\oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ are unitary equivalent, there exists a unitary operator U such that $U\mathcal{A}_{2n}^{(2)}U^* = \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$. Let $\varphi : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \rightarrow \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ be a map defined by $\varphi(A) = U\phi(U^*AU)U^*$ for all A in $\oplus_{i=1}^{k} \mathcal{A}_{2n_i}$. Then φ is an isometry and the diagram

$$\begin{array}{ccc}
\mathcal{A}_{2n}^{(2)} & \xrightarrow{\ \phi\ } & \mathcal{A}_{2n}^{(2)} \\
\uparrow & & \downarrow \\
\oplus_{i=1}^{k} \mathcal{A}_{2n_i} & \xrightarrow{\ \varphi\ } & \oplus_{i=1}^{k} \mathcal{A}_{2n_i}
\end{array}$$

commutes. Hence in this section, we investigate isometric maps φ from $\oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ to $\oplus_{i=1}^{k} \mathcal{A}_{2n_i}$.

Since $\mathcal{L}$ is a commutative subspace lattice, from the Lemmas 1 and 2 in [3], we have the following theorem.

THEOREM 4.1. *Let $\varphi : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \rightarrow \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ be an isometry. Then $\varphi(I)$ is a diagonal unitary operator.*

Let $\varphi : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \rightarrow \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ be an isometry and let $\varphi(I) = U$. Define $\tilde{\varphi} : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \rightarrow \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ by $\tilde{\varphi}(A) = U^*\varphi(A)$ for every A in

$\oplus_{i=1}^{k}\mathcal{A}_{2n_i}$. Then $\tilde{\varphi}$ is an isometry such that $\tilde{\varphi}(I) = I$. Since the main theorem would be true of φ if it were true of $\tilde{\varphi}$, we remark that all isometries in this paper carry the identity into the identity. Let $\mathcal{D} = \{A :$ A is a diagonal operator in $\oplus_{i=1}^{k}\mathcal{A}_{2n_i}\}$. Then it is easy to check that $\mathcal{D}$ is the smallest von Neumann algebra containing $\mathcal{L}$ and $\mathcal{D} = (\oplus_{i=1}^{k}\mathcal{A}_{2n_i}) \cap (\oplus_{i=1}^{k}\mathcal{A}_{2n_i})^*$, where $(\oplus_{i=1}^{k}\mathcal{A}_{2n_i})^* = \{A^* : A$ is in $\oplus_{i=1}^{k}\mathcal{A}_{2n_i}\}$. From the Lemmas 5 and 7 and Definition 6 in [3], we have the following Theorem.

THEOREM 4.2. *Let $\varphi : \oplus_{i=1}^{k}\mathcal{A}_{2n_i} \to \oplus_{i=1}^{k}\mathcal{A}_{2n_i}$ be a surjective isometry such that $\varphi(I) = I$. Then*

(1) $\varphi(\mathcal{D}) = \mathcal{D}$.
(2) *E is a projection in $\mathcal{D}$ if and only if $\varphi(E)$ is a projection in $\mathcal{D}$.*

From Lemma 11 in [3] and the minimal properties of projections, we can prove the following theorem.

THEOREM 4.3. *Let $\varphi : \oplus_{i=1}^{k}\mathcal{A}_{2n_i} \to \oplus_{i=1}^{k}\mathcal{A}_{2n_i}$ be an isometry such that $\varphi(I) = I$. Then $\varphi(E_{ii})$ is a rank one operator for each $i = 1, 2, \cdots, 2n$.*

From the modification of the Lemma 15 in [3], we have the following lemma.

LEMMA 4.4. *Let $\varphi : \oplus_{i=1}^{k}\mathcal{A}_{2n_i} \to \oplus_{i=1}^{k}\mathcal{A}_{2n_i}$ be an isometry such that $\varphi(I) = I$. Let E be a projection in $\mathcal{D}$ and let T be in $\oplus_{i=1}^{k}\mathcal{A}_{2n_i}$ with $T = ETE^{\perp}$. Then we have*

$$\varphi(T) = \varphi(E)\varphi(T)\varphi(E)^{\perp} + \varphi(E)^{\perp}\varphi(T)\varphi(E).$$

Note that $E_{ii} = [e_i]$ for all $i = 1, 2, \cdots, 2n$. From Lemma 4.4, we get the following theorem.

THEOREM 4.5. *Let $\varphi : \oplus_{i=1}^{k}\mathcal{A}_{2n_i} \to \oplus_{i=1}^{k}\mathcal{A}_{2n_i}$ be an isometry such that $\varphi(I) = I$. Let $\varphi(E_{2j-1,2j-1}) = E_{pp}$ and $\varphi(E_{2j,2j}) = E_{qq}$. Then $|p - q| = 1$ or $2n_i - 1$ for some $i = 1, 2, \cdots, k$ and $\varphi(E_{2j-1,2j}) = \alpha_{pq}E_{pq}$ or $\alpha_{qp}E_{qp}$ for some complex number α_{pq} or α_{qp}.*

PROOF. Since

$$E_{2j,2j}^{\perp}E_{2j-1,2j}E_{2j,2j} = E_{2j-1,2j}$$

and

$$E_{2j-1,2j-1}E_{2j-1,2j}E_{2j-1,2j-1}^{\perp} = E_{2j-1,2j},$$

Lemma 4.4 tells us that

(a) $$\varphi(E_{2j-1,2j}) = E_{qq}^{\perp}\varphi(E_{2j-1,2j})E_{qq} + E_{qq}\varphi(E_{2j-1,2j})E_{qq}^{\perp}$$

and

(b) $$\varphi(E_{2j-1,2j}) = E_{pp}\varphi(E_{2j-1,2j})E_{pp}^{\perp} + E_{pp}^{\perp}\varphi(E_{2j-1,2j})E_{pp}$$

From the equation (a) we can get the following:

(1) If $q = 1$, then $\varphi(E_{2j-1,2j}) = \alpha_{12}E_{12} + \alpha_{1,2n_1}E_{1,2n_1}$ for some complex numbers α_{12} and $\alpha_{1,2n_1}$.

(2) If $q = \sum_{i=1}^{m} 2n_i + 1$ for some $m(m = 1, 2, \cdots, k-1)$, then $\varphi(E_{2j-1,2j}) = \alpha_{q,q+1}E_{q,q+1} + \alpha_{q,q+2n_{m+1}-1}E_{q,q+2n_{m+1}-1}$ for some complex numbers $\alpha_{q,q+1}$ and $\alpha_{q,q+2n_{m+1}-1}$.

(3) If $q \neq 1$, $q \neq \sum_{i=1}^{m} 2n_i + 1$ for all $m(m = 1, 2, \cdots, k-1)$ and q is an odd number , then $\varphi(E_{2j-1,2j}) = \alpha_{q,q-1}E_{q,q-1} + \alpha_{q,q+1}E_{q,q+1}$ for some complex numbers $\alpha_{q,q-1}$ and $\alpha_{q,q+1}$.

(4) If $q \neq \sum_{i=1}^{m} 2n_i$ for all $m(m = 1, 2, \cdots, k)$ and q is an even number, then $\varphi(E_{2j-1,2j}) = \alpha_{q-1,q}E_{q-1,q} + \alpha_{q+1,q}E_{q+1,q}$ for some complex numbers $\alpha_{q-1,q}$ and $\alpha_{q+1,q}$.

(5) If $q = \sum_{i=1}^{m} 2n_i$ for some $m = 1, 2, \ldots, k$, then $\varphi(E_{2j-1,2j}) = \alpha_{q-1,q}E_{q-1,q} + \alpha_{q-2n_m+1,q}E_{q-2n_m+1,q}$ for some complex numbers $\alpha_{q-1,q}$ and $\alpha_{q-2n_m+1,q}$.

From the equation (b) we can get the following:

(1) If $p = 1$, then $\varphi(E_{2j-1,2j}) = \alpha_{12}E_{12} + \alpha_{1,2n_1}E_{1,2n_1}$ for some complex numbers α_{12} and $\alpha_{1,2n_1}$.

(2) If $p = \sum_{i=1}^{m} 2n_i + 1$ for some $m(m = 1, 2, \cdots, k-1)$, then $\varphi(E_{2j-1,2j}) = \alpha_{p,p+1}E_{p,p+1} + \alpha_{p,p+2n_{m+1}-1}E_{p,p+2n_{m+1}-1}$ for some complex numbers $\alpha_{p,p+1}$ and $\alpha_{p,p+2n_{m+1}-1}$.

(3) If $p \neq 1$, $p \neq \sum_{i=1}^{m} 2n_i + 1$ for all $m(m = 1, 2, \cdots, k-1)$ and p is an odd number, then $\varphi(E_{2j-1,2j}) = \alpha_{p,p-1}E_{p,p-1} + \alpha_{p,p+1}E_{p,p+1}$ for some complex numbers $\alpha_{p,p-1}$ and $\alpha_{p,p+1}$.

(4) If $p \neq \sum_{i=1}^{m} 2n_i$ for all $m(m = 1, 2, \cdots, k)$ and p is an even number, then $\varphi(E_{2j-1,2j}) = \alpha_{p-1,p}E_{p-1,p} + \alpha_{p+1,p}E_{p+1,p}$ for some complex numbers $\alpha_{p-1,p}$ and $\alpha_{p+1,p}$.

(5) If $p = \sum_{i=1}^{m} 2n_i$ for some $m = 1, 2, \ldots, k$, then $\varphi(E_{2j-1,2j}) = \alpha_{p-1,p}E_{p-1,p} + \alpha_{p-2n_m+1,p}E_{p-2n_m+1,p}$ for some complex numbers $\alpha_{p-1,p}$ and $\alpha_{p-2n_m+1,p}$.

Hence we have the following conclusion.

If $p = 1$, then $q = 2$ or $q = 2n_1$ and $\varphi(E_{2j-1,2j}) = \alpha_{pq}E_{pq}$ for some complex number α_{pq}.

If $p = \sum_{i=1}^{m} 2n_i+1$ for some $m(1 \le m \le k-1)$, then $q = \sum_{i=1}^{m} 2n_i+2$ or $q = \sum_{i=1}^{m+1} 2n_i$ and $\varphi(E_{2j-1,2j}) = \alpha_{pq}E_{pq}$ for some complex number α_{pq}.

If $1 \le p \le 2n_1$ or $\sum_{i=1}^{m} 2n_i + 1 < p < \sum_{i=1}^{m+1} 2n_i$ and p is even, then $q = p-1$ or $q = p+1$ and $\varphi(E_{2j-1,2j}) = \alpha_{qp}E_{qp}$ for some complex number α_{qp}.

If $1 \le p \le 2n_1$ or $\sum_{i=1}^{m} 2n_i + 1 < p < \sum_{i=1}^{m+1} 2n_i$ and p is odd, then $q = p-1$ or $q = p+1$ and $\varphi(E_{2j-1,2j}) = \alpha_{pq}E_{pq}$ for some complex number α_{pq}.

If $p = 2n_1$, then $q = 1$ or $q = 2n_1 - 1$ and $\varphi(E_{2j-1,2j}) = \alpha_{qp}E_{qp}$ for some complex number α_{qp}.

If $p = \sum_{i=1}^{m} 2n_i$ for some $m(2 \le m \le k)$, then $q = \sum_{i=1}^{m-1} 2n_i + 1$ or $q = \sum_{i=1}^{m} 2n_i - 1$ and $\varphi(E_{2j-1,2j}) = \alpha_{qp}E_{qp}$ for some complex number α_{qp}.

From Theorem 4.5, we can get the following corollary.

COROLLARY 4.6. *Let $\varphi : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \to \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ be an isometry such that $\varphi(I) = I$. Let $\varphi(E_{2j-1,2j-1}) = E_{pp}$ and $\varphi(E_{2j,2j}) = E_{qq}$. Then $\sum_{i=1}^{m-1} 2n_i + 1 \le p \le \sum_{i=1}^{m} 2n_i$ if and only if $\sum_{i=1}^{m-1} 2n_i + 1 \le q \le \sum_{i=1}^{m} 2n_i$*

By an argument similar to that of Theorem 4.5, we can obtain the following theorem.

THEOREM 4.7. *Let $\varphi : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \to \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ be an isometry such that $\varphi(I) = I$. Let $\varphi(E_{2j,2j}) = E_{pp}$ and $\varphi(E_{2j+1,2j+1}) = E_{qq}$.*

(1) *If $j \ne \sum_{i=1}^{m} n_i$ for all $m = 1, 2, \cdots, k-1$, then $|p - q| = 1$ or $2n_i - 1$ for some $i = 1, 2, \cdots, k$ and $\sum_{i=1}^{m-1} 2n_i + 1 \le p \le \sum_{i=1}^{m} 2n_i$ if and only if $\sum_{i=1}^{m-1} 2n_i + 1 \le q \le \sum_{i=1}^{m} 2n_i$.*

(2) *If* $j = \sum_{i=1}^{m} n_i$ *for some* $m = 1, 2, \cdots, k-1$ *and if* $\sum_{i=1}^{m-1} 2n_i + 1 \le p \le \sum_{i=1}^{m} 2n_i$, *then* $\sum_{i=1}^{m} 2n_i + 1 \le q \le \sum_{i=1}^{m+1} 2n_i$.

Let $F_{2n_i} = \sum_{j=2n_1+2n_2+\cdots+2n_{i-1}+1}^{2n_1+2n_2+\cdots+2n_i} E_{jj}$ and let $A'_{2n_i} = F_{2n_i}(\oplus_{j=1}^{k} A_{2n_j})$ for all $i = 1, 2, \cdots k$. Let $\mathcal{A}'_{2n_i} = \{F_{2n_i}(\oplus_{j=1}^{k} A_{2n_j}) : (\oplus_{j=1}^{k} A_{2n_j}) \in \oplus_{j=1}^{k} \mathcal{A}_{2n_j}\}$. From Theorem 4.4 and Corollary 4.5 and Theorem 4.6, we can get the following Theorem.

THEOREM 4.8. *Let* $\varphi : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \to \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ *be an isometry such that* $\varphi(I) = I$ *and let* $n_i \ne n_j$ *for all* $i, j (1 \le i, j \le k)$. *Then* $\varphi(\mathcal{A}'_{2n_i}) = \mathcal{A}'_{2n_i}$ *for all* $i = 1, 2, \cdots, k$.

THEOREM 4.9. *Let* $\varphi_i : \mathcal{A}_{2n_i} \to \mathcal{A}_{2n_i}$ *be an isometry for all* $i = 1, 2, \cdots, k$. *Then the map* $\varphi : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \to \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ *defined by* $\varphi(\oplus_{i=1}^{k} A_i) = \oplus_{i=1}^{k} \varphi_i(A_i)$ *is an isometry.*

PROOF. Suppose $\varphi_i : \mathcal{A}_{2n_i} \to \mathcal{A}_{2n_i}$ is an isometry for all $i = 1, 2, \cdots, k$. Then $\|\varphi(\oplus_{i=1}^{k} A_i)\| = \| \oplus_{i=1}^{k} \varphi_i(A_i)\| = max\{\|\varphi_i(A_i)\| : i = 1, 2, \cdots, k\} = max\{\|A_i\| : i = 1, 2, \cdots, k\} = \| \oplus_{i=1}^{k} A_i\|$. Hence $\varphi : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \to \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ is an isometry.

THEOREM 4.10. *Let* $\varphi : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \to \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ *be an isometry such that* $\varphi(I) = I$ *and let* $n_1 = n_2 = \cdots = n_k$. *Then there exist isometries* $\varphi_i : \mathcal{A}_{2n_i} \to \mathcal{A}_{2n_i}$ *for all* $i = 1, 2, \cdots, k$ *such that* $\varphi = \oplus_{i=1}^{k} \varphi_i$.

PROOF. Suppose $\varphi(\mathcal{A}'_{2n_i}) = \mathcal{A}'_{2n_{\sigma(i)}}$ for all $i = 1, 2, \cdots, k$, where

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{pmatrix}$$

is a permutation. Let $\varphi_i = \varphi|_{\mathcal{A}'_{2n_i}}$ for all $i = 1, 2, \cdots, k$. Then $\varphi_i : \mathcal{A}_{2n_i} \to \mathcal{A}_{2n_{\sigma(i)}} = \mathcal{A}_{2n_i}$ is an isometry and $\varphi = \oplus_{i=1}^{k} \varphi_i$.

THEOREM 4.11. *Let* $\varphi : \oplus_{i=1}^{k} \mathcal{A}_{2n_i} \to \oplus_{i=1}^{k} \mathcal{A}_{2n_i}$ *be an isometry such that* $\varphi(I) = I$ *and let* $n_i \ne n_j$ *for all* $i, j (1 \le i, j \le k)$. *Then there exist isometries* $\varphi_i : \mathcal{A}_{2n_i} \to \mathcal{A}_{2n_i}$ *for all* $i = 1, 2, \cdots, k$ *such that* $\varphi = \oplus_{i=1}^{k} \varphi_i$.

PROOF. Suppose $\varphi : \oplus_{i=1}^k \mathcal{A}_{2n_i} \to \oplus_{i=1}^k \mathcal{A}_{2n_i}$ is an isometry such that $\varphi(I) = I$. Let $\varphi_i = \varphi|_{\mathcal{A'}_{2n_i}}$ for all $i = 1, 2, \cdot, k$. Then for each $i(1 \leq i \leq k)$, $\varphi_i : \mathcal{A}_{2n_i} \to \mathcal{A}_{2n_i}$ is an isometry and $\varphi = \oplus_{i=1}^k \varphi_i$.

From Theorems 2.5, 4.10 and 4.11, we can get the following Theorem.

THEOREM 4.12. *Let $\varphi : \oplus_{i=1}^k \mathcal{A}_{2n_i} \to \oplus_{i=1}^k \mathcal{A}_{2n_i}$ be an isometry such that $\varphi(I) = I$. Then there exist unitary operators U_i for all $i = 1, 2, \cdots, k$ such that $\varphi(\oplus_{i=1}^k A_i) = \oplus_{i=1}^k U_i B_i U_i^*$, where $B_i = A_i$ or $B_i = A_i^t$ for all A_i in $\mathcal{A}_{2n_i}$.*

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