Comm. Korean Math. Soc. **10** (1995), No. 3, pp. 575–581

QUANTUM GROUP $X_q(2)$

SEI-QWON OH AND YONGYEON SHIN

ABSTRACT. The simple modules and the simple comodules of the quantum group $X_q(2)$, defined by M. L. Ge, N. H. Jing and Y. S. Wu, are classified.

0. Introduction

Recently, the so-called quantum groups have attracted a lot of attention to non-commutative algebraists because they are expected to exhibit a structure similar to that of enveloping algebras of certain Lie algebras. The purpose of this paper is to support this philosophy by characterizing the simple modules, the primitive ideals and the simple comodules of a kind of quantum group $X_q(2)$, which is constructed from a nonstandard braid group representation by employing the Faddeev-Reshetikhin-Takhtajan constructive method by N. H. Jing, M. L. Ge and Y. S. Wu in [**1**]. (See [**6, 9**] for additional evidence.)

On the other hand, in [**4**], D. A. Jordan studied simple modules of a class of iterated skew polynomial rings $R(A, \alpha, u)$ in two indeterminates over a commutative ring A. This class contains the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, the quantum enveloping algebra $U_q(\mathfrak{sl}(2, \mathbb{C}))$, the quantum matrices $\mathcal{O}_q(M(2, \mathbb{C}))$, the algebras similar to $U(\mathfrak{sl}(2, \mathbb{C}))$ considered in [**8**] and a ring introduced by Podles [**7**]. Since the quantum group $X_q(2)$ is defined by a factor of a quantum group $\hat{X}_q(2)$ to a Hopf ideal and $\hat{X}_q(2)$ is shown to be of this class $R(A, \alpha, u)$, we review D. A. Jordan's construction in [**4**] in the first section. In the second section, we show that:

Received February 20, 1995. Revised May 11, 1995.
1991 AMS Subject Classification: 16D30, 16D60.
Key words: skew polynomial ring, simple module, simple comodule.
The first author was partially supported by KOSEF Grant 931-0100-018-2

(1) There exists a bijective map from the set $\mathbb{C}^* \times \mathbb{C}^*$ onto the set of all simple right $X_q(2)$-modules up to isomorphic. In particular, every point of the set $T = \{(\lambda, \pm\lambda^{-1}) \mid \lambda \in \mathbb{C}^*\}$ corresponds to one dimensional simple modules and every point of the set $\mathbb{C}^* \times \mathbb{C}^* \backslash T$ corresponds to two dimensional ones.

(2) $X_q(2)$ is a prime algebra but not an integral domain.

In final section, we show that:

(3) The simple comodules of $X_q(2)$ are of the form $\mathbb{C}k^n\xi^m$, $(n, m) \in \mathbb{Z} \times \mathbb{Z}$.

The ring theoretical terminologies and definitions are refered to [**5**], and the definitions of comodule and Hopf algebras are refered to [**2**, **10**].

1. D. A. Jordan's construction

1.1 Given a commutative integral domain A, an automomorphm α of A and an element u of A, the ring $R = R(A, \alpha, u)$ is the iterated skew polynomial ring $R = A[y, \alpha][x, \beta, \delta]$, where β is the extension of α^{-1} in $A[y, \alpha]$ by setting $\beta(y) = y$, and β-derivation δ is defined by $\delta(A) = 0, \delta(y) = u - \alpha(u)$ (see [**4**, 1.1]).

1.2 Let $\lambda \in A$ be such that $\alpha(\lambda) = \lambda$ and let M be a maximal ideal of A containing $u - \lambda$. We use $V(M, \lambda)$ to denote the right A-module $\bigoplus_{i \geq 0}(A/\alpha^{-i}(M))$. We give a right R-module structure on $V(M, \lambda)$ as follows (see [**4**, 3.1]) :

$$(a + \alpha^{-i}(M))y = \alpha^{-1}(a) + \alpha^{-(i+1)}(M)$$
$$(a + \alpha^{-i}(M))x = \alpha(a)(\alpha(u) - \lambda) + \alpha^{-(i-1)}(M).$$

Let $j \geq 0$ be the least number such that $u - \lambda \in \alpha^{-j}(M)$, equivalently $\alpha^j(u) - u \in M$. Then $\oplus_{i \geq j} A/\alpha^{-i}(M)$ is a maximal submodule of $V(M, \lambda)$. The corresponding factor, denoted by $L(M, \lambda)$, is a simple right R-module.

The modules $V(M, \lambda)$ and $L(M, \lambda)$ are prototypes of the Verma modules of $U(\mathfrak{sl}(2))$.

2. Quantum groups $\hat{X}_q(2)$ and $X_q(2)$

DEFINITION 2.1. (see [1]) A Hopf algebra $\hat{X}_q(2) = \mathbb{C}[x, y, k^{\pm 1}, \xi^{\pm 1}]$ has the following relations:

$$\begin{aligned} &[k,\xi] = 0, && [x,y] = (k\xi - k^{-1}\xi^{-1})/(q - q^{-1}) \\ &kxk^{-1} = qx, && \xi x \xi^{-1} = -q^{-1}x \\ &kyk^{-1} = q^{-1}y, && \xi y \xi^{-1} = -qy \\ &\Delta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1}, && \Delta(\xi^{\pm 1}) = \xi^{\pm 1} \otimes \xi^{\pm 1} \\ &\Delta(x) = x \otimes k + \xi^{-1} \otimes x, && \Delta(y) = y \otimes \xi + k^{-1} \otimes y \\ &\varepsilon(x) = \varepsilon(y) = 0, && \varepsilon(k) = \varepsilon(\xi) = 1 \\ &S(k) = k^{-1}, && S(\xi) = \xi^{-1} \\ &S(x) = -\xi x k^{-1}, && S(y) = -ky\xi^{-1} \end{aligned}$$

Let $I = \langle x^2, y^2 \rangle$. Then $\Delta(I) \subseteq I \otimes \hat{X}_q(2) + \hat{X}_q(2) \otimes I$ and $S(I) \subseteq I$. Hence I is a Hopf ideal of $\hat{X}_q(2)$. Set $X_q(2) = \hat{X}_q(2)/I$. Then $X_q(2)$ also becomes a Hopf algebra.

2.2 Let A be the subalgebra of $\hat{X}_q(2)$ generated by $k^{\pm 1}$ and $\xi^{\pm 1}$, α an automorphism of the commutative integral domain A defined by $\alpha(k) = qk, \alpha(\xi) = -q^{-1}\xi$ and let $u = -\frac{k\xi - k^{-1}\xi^{-1}}{2(q-q^{-1})}$. We see immediately that $\hat{X}_q(2) = R(A, \alpha, u)$. Hence $\hat{X}_q(2)$ is noetherian, integral domain, the Gelfand-Kirillov dimension 4 and has a $\mathbb{C}$-basis $\{k^i \xi^j y^m x^n \mid i, j \in \mathbb{Z}, m, n \in \mathbb{Z}_+\}$.

THEOREM 2.3. *There exists a bijective map from the set $\mathbb{C}^* \times \mathbb{C}^*$ onto the set of all simple right $X_q(2)$-modules up to isomorphic. In particular, every point of the set $T = \{(\lambda, \pm\lambda^{-1}) \mid \lambda \in \mathbb{C}^*\}$ corresponds to one dimensional simple modules and every point of the set $\mathbb{C}^* \times \mathbb{C}^* \backslash T$ corresponds to two dimensional ones.*

PROOF. Notice that every maximal ideal of A is of the form $M_{\lambda,\mu} = \langle k - \lambda, \xi - \mu \rangle$ for some $\lambda, \mu \in \mathbb{C}^*$ and, for $d \geq 1$,

$$u - \alpha^d(u) = \begin{cases} 2u & d \text{ is odd} \\ 0 & d \text{ is even.} \end{cases}$$

Let X be a simple right $X_q(2)$-module. Then X is a simple $\hat{X}_q(2)$-module annihilated by x^2 and y^2. By [**4**, 3.10], X is isomorphic to $L(M_{\lambda,\mu})$ and the dimension d of X is the least positive integer d such that $\alpha^d(u) - u \in M_{\lambda,\mu}$. Moreover $L(M_{\lambda,\mu})$ is isomorphic to $L(M_{\lambda',\mu'})$ if and only if $(\lambda,\mu) = (\lambda',\mu')$ by the translation of the proof of [**3**, 20.3 Theorem A]. Since $d \le 2$, the simple $\hat{X}_q(2)$-module $L(M_{\lambda,\mu})$ is annihilated by x^2 and y^2 by 1.2. Therefore $L(M_{\lambda,\mu})$ is simple $X_q(2)$-module and

$$\dim_{\mathbb{C}}(X) = \dim(L(M_{\lambda,\mu})) = \begin{cases} 1 & u \in M_{\lambda,\mu} \\ 2 & u \notin M_{\lambda,\mu}. \end{cases}$$

From the fact that $u \in M_{\lambda,\mu}$ if and only if $\lambda^2\mu^2 = 1$, each point of T corresponds to two non-isomorphic simple modules $L(M_{\lambda,\pm\lambda^{-1}})$ with dimension 1. □

PROPOSITION 2.4. *$X_q(2)$ is a prime algebra but not an integral domain.*

PROOF. Clearly, $X_q(2)$ is not an integral domain because $\langle x^2, y^2\rangle$ is not a completely prime ideal of $\hat{X}_q(2)$. Any element f of $X_q(2)$ is uniquely expressed as

$$f = \alpha_1 x + \alpha_2 y + \alpha_3 xy + \alpha_4$$

where $\alpha_i \in \mathbb{C}[k^{\pm 1}, \xi^{\pm 1}]$ by 2.2. Let $f = \alpha_1 x + \alpha_2 y + \alpha_3 xy + \alpha_4$ and $g = \beta_1 x + \beta_2 y + \beta_3 xy + \beta_4$ be any nonzero elements of $X_q(2)$. The fact that $fX_q(2)g \neq 0$ is proved by straight calculations. □

3. Simple comodule of $X_q(2)$

LEMMA 3.1. *Let A be a vector space and B a subspace of A, $\{x_i\}$ a basis of A and $y_i \in A$. If $\sum y_i \otimes x_i \in B \otimes A$, then $y_i \in B$ for all i.*

PROOF. It is clear from the natural basis of $A \otimes A$. □

THEOREM 3.2. *The simple comodules of* $X_q(2)$ *are of the form* $\mathbb{C}k^n\xi^m$, $(n,m) \in \mathbb{Z} \times \mathbb{Z}$.

PROOF. Let I be an ideal generated by x and y. Then I is a Hopf ideal and $X_q(2)/I$ is cocommutative Hopf algebra. Let $\pi : X_q(2) \longrightarrow X_q(2)/I$ be the natural map and let V be a simple sub-comodule of $X_q(2)$. Then $\tau = 1 \otimes \pi \circ \Delta : V \longrightarrow V \otimes [X_q(2)/I]$ is a comodule structure map. Now we show that

$$V_{(i,j)} \subset V$$

where $V_{(i,j)} = \{f \in V | f = \alpha k^i \xi^j + \beta k^{i-1}\xi^j x + \gamma k^i \xi^{j-1} y + \delta k^{i-1}\xi^{j-1}xy\}$. Since V is an $X_q(2)/I$-comodule and $X_q(2)/I \simeq \mathbb{C}[k^{\pm 1}, \xi^{\pm 1}]$, we can think V as a $\mathbb{C}[k^{\pm 1}, \xi^{\pm 1}]$-comodule. Let us put $G = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[k^{\pm 1}, \xi^{\pm 1}], \mathbb{C})$ and define the module structure $G \times V \longrightarrow V$ as $g \cdot v = (1 \otimes g)(1 \otimes \pi)\Delta(v)$ for any $g \in G, v \in V$. Now we define the algebra homomorphisms $g_{(\alpha,\beta)} \in G$ as follows:

$$g_{(\alpha,\beta)}(k) = \alpha, \quad g_{(\alpha,\beta)}(\xi) = \beta$$

Since $V \subseteq X_q(2)$, v can be expressed as the form $\sum c_{ijlm} x^i y^j k^l \xi^m$ for any v in V where $i = 0,1, j = 0,1, l, m \in \mathbb{Z}$. Then

$$\begin{aligned}\tau(v) &= (1 \otimes \pi)\Delta(v) \\ &= \sum c_{ijlm}(x^i \otimes k^i)(y^j \otimes \xi^j)(k^l \otimes k^l)(\xi^m \otimes \xi^m) \\ &= \sum c_{ijlm}(x^i y^j k^l \xi^m \otimes k^{i+l}\xi^{j+m}).\end{aligned}$$

Thus

$$\begin{aligned}g_{(1,1)} \cdot v &= \sum c_{ijlm} x^i y^j k^l \xi^m \\ g_{(1,2)} \cdot v &= \sum c_{ijlm} 2^{j+m} x^i y^j k^l \xi^m.\end{aligned}$$

Therefore $V = \oplus V_{(i,j)}$ where $V_{(i,j)} = \{f \in V | f = \alpha k^i \xi^j + \beta k^{i-1}\xi^j x + \gamma k^i \xi^{j-1} y + \delta k^{i-1}\xi^{j-1}xy\}$ which is said to be weight space with weight (i,j). Let $V_{(n,m)}$ be the minimal weight space. Choose $0 \neq f \in V_{(n,m)}$ and put

$$f = \alpha k^n \xi^m + \beta k^{n-1}\xi^m x + \gamma k^n \xi^{m-1} y + \delta k^{n-1}\xi^{m-1} xy$$

$$\Delta(f) \in V \otimes X_q(2)$$

and

$$\begin{aligned}\Delta(f) =&\alpha k^n\xi^m \otimes k^n\xi^m + (\beta k^{n-1}\xi^m \otimes k^{n-1}\xi^m)(x \otimes k + \xi^{-1} \otimes x)\\ &+ (\gamma k^n\xi^{m-1} \otimes k^n\xi^{m-1})(y \otimes \xi + k^{-1} \otimes y)\\ &+ (\delta k^{n-1}\xi^{m-1} \otimes k^{n-1}\xi^{m-1})(x \otimes k + \xi^{-1} \otimes x)(y \otimes \xi + k^{-1} \otimes y)\\ =&(\alpha k^n\xi^m + \beta k^{n-1}\xi^m x + \gamma k^n\xi^{m-1}y + \delta k^{n-1}\xi^{m-1}xy) \otimes k^n\xi^m\\ &+ (\beta k^{n-1}\xi^{m-1} - q\delta k^{n-1}\xi^{m-2}y) \otimes k^{n-1}\xi^m x\\ &+ (\gamma k^{n-1}\xi^{m-1} + q\delta k^{n-2}\xi^{m-1}x) \otimes k^n\xi^{m-1}y\\ &+ \delta k^{n-2}\xi^{m-2} \otimes k^{n-1}\xi^{m-1}xy.\end{aligned}$$

By 3.1, $\delta k^{n-2}\xi^{m-2} \in V$. If δ is nonzero then $\delta k^{n-2}\xi^{m-2}$ is in $V_{(n-2,m-2)}$. This contradicts to the fact that $V_{(n,m)}$ is the minimal weight space. Hence δ is zero. Similarly γ and β are zero also. So $f = \alpha k^n\xi^m \in V$. Thus $\mathbb{C}k^n\xi^m$ is a subcomodule of V and $\mathbb{C}k^n\xi^m = V$ by simplicity.

References

1. M. L. Ge, N. H. Jing, Y. S. Wu, *A new quantum group associated with a non-standard braid group representation*, Letters in Mathematical Physics **21** (1991), 193-203.
2. J. A. Green, *Locally finite representations*, Journal of algebra **41** (1976), 137-171.
3. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, Graduate texts in mathematics Vol. 9, 1972.
4. D. A. Jordan, *Iterated skew polynomial rings and quantum groups*, Journal of algebra **156** (1993), 194-218.
5. J. C. McConnel, J. C. Robson, *Noncommutative Noetherian Rings*, Wiley, 1987.
6. Sei-Qwon Oh, *Primitive ideals of the coordinate ring of quantum symplectic space*, Journal of algebra (to appear).
7. P. Podles, *Quantum spheres*, Letters in Math. Phys **14** (1987), 193-202.
8. S. P. Smith, *A class of algebras similar to the enveloping algebra of* $\mathfrak{sl}(2)$, Trans. Amer. Math. Soc **322** (1990), 285-314.
9. ———, *Quantum Groups, An introduction and survey for ring theorists*, in Noncommutative Rings (S. Montgomery and P. F. Smith, eds.), M. S. R. I. Publ. 24, Springer-Verlag, New York, 1992, pp. 131-178.
10. M. Sweedler, *Hopf algebras*, W. A. Benjamin, 1969.

Sei-Qwon Oh
Department of Mathematics
Chungnam National University
Taejon 305-764, Korea
e-mail: sqoh@math.chungnam.ac.kr

Yongyeon Shin
Korea Atomic Energy Research Institute
Taejon 305-353, Korea