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THE ALLOWANCE OF IDEMPOTENT OF SIGN PATTERN MATRICES

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ABSTRACT. A matrix whose entries consist of the symbols +, − and 0 is called a *sign pattern matrix*. In [1], a graph theoretic characterization of sign idempotent pattern matrices was given. A question was given for the sign patterns which allow idempotence. We characterized the sign patterns which allow idempotence in the sign idempotent pattern matrices.

0. Introduction

A matrix whose entries consist of the symbols +, −, and 0 is called a *sign pattern matrix*. For a real matrix B, by *sgn* B we mean the sign pattern matrix in which each positive (respectively, negative, zero) entry is replaced by + (respectively, $-, 0$). For each n-by-n sign pattern matrix $\boldsymbol{A}$, there is a natural class of real matrices whose entries have the signs indicated by $\boldsymbol{A}$. If $\boldsymbol{A} = (a_{ij})$ is an n-by-n sign pattern matrix, then the *sign pattern class of* $\boldsymbol{A}$ is defined by

$$Q(\boldsymbol{A}) = \{B \in M_n(R) | sgnB = \boldsymbol{A}.\}$$

Recall that a real n-by-n matrix B is said to be idempotent if $B = B^2$. Analogously, a squre sign pattern matrix $\boldsymbol{A}$ is said to be *sign idempotent* if $B^2 \in Q(\boldsymbol{A})$ whenever $B \in Q(\boldsymbol{A})$; henceforth we write $\boldsymbol{A} = \boldsymbol{A}^2$.

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If $\boldsymbol{A}$ and $\boldsymbol{C}$ are n-by-n sign pattern matrices, then $\boldsymbol{A}+\boldsymbol{C}$ exists, that is, $\boldsymbol{A}+\boldsymbol{C}$ is qualitatively defined if $a_{ij}c_{ij} \neq -$ for all i and j in $\{1, 2, \cdots, n\}$. The product $\boldsymbol{AC}$ exists if no two terms in the sum

$$\sum_{k=1}^{n} a_{ik}c_{kj}$$

are oppositely signed for all i and j in $\{1, 2, \cdots, n\}$.

Suppose $\mathcal{P}$ is a property a real matrix may or may not have. A sign pattern matrix $\boldsymbol{A}$ is said to *require* $\mathcal{P}$ if *every* real matrix in $Q(\boldsymbol{A})$ has the property $\mathcal{P}$. Also, a sign pattern matrix $\boldsymbol{A}$ is said to ***allow*** $\mathcal{P}$ if *some* real matrix in $Q(\boldsymbol{A})$ has property $\mathcal{P}$. These definitions raise the following questions(*See*.[1]):

(a) Identifying the sign idempotent sign patterns.

(b) Identifying the arbitrary sign patterns that allow idempotence.

First we will introduce some examples, the following sign pattern matrix

(1) $$\boldsymbol{A} = \begin{pmatrix} + & - \\ 0 & + \end{pmatrix} \text{ is sign idempotent,}$$

but, no real matrix in the sign pattern class of $\boldsymbol{A}$ is idempotent. This implies that $\boldsymbol{A}$ does not allow idempotent.

The other matrix

(2) $$B = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ is also real idempotent,}$$

but sign pattern matrix $\boldsymbol{B}$ is not sign idempotent.

We now characterize the sign patterns that allow idempotence. In order to simplify our notation, in the remainder of this paper, we let the index set $\{1, 2, \cdots, n\}$ be represented by I_n, SI be the class of sign idempotent matrices, and $\boldsymbol{P} = (p_{ij})$ be the product matrix $\boldsymbol{A}^2$.

LEMMA 1.1. *If* $A = [a_{ij}] \in SI$, *then* $a_{ii} = 0$ *or* $+$ *for all* $i \in I_n$.

PROOF. Since $A^2 = A$, let $[p_{ij}] = A^2$ then $p_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$. Therefore

$$p_{ii} = a_{i1}a_{1i} + a_{i2}a_{2i} + ... + a_{ii}a_{ii} + ... + a_{in}a_{ni} = a_{ii}.$$

Since $\sum_{k=1}^{n} a_{ik}a_{kj}$ contains $a_{ii}a_{ii}$, from the definition of product, all term must have the same sign or have some zeros in it. Thus, without loss of genernality, $a_{ii}^2 = a_{ii}$. So, $a_{ii} = +$ or 0 for all $i \in I_n$. ∎

A sign pattern matrix A is called *partly decomposable* if there are n-by-n permutation matrices Q_1 and Q_2 such that

$$Q_1 A Q_2^T = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square matrices, In the special case, when $Q_2 = Q_1^T$, then A is said to be *reducible.* If no such permutation matrices exist, then A is said to be *(fully) indecomposable.*

Here we consider the two cases, the one is an *irreducible* sign idempotent case and the other is a reducible case.

1. Irreducible sign pattern matrices

First, we consider the case that A is an irreducible sign idempotent matrix. We may note that if $a_{ij} = 0$ for some indices i and j in I_n, then $A = A^2$ only if A is *partly decomposable.* ([1]). This imply that every irreducible sign idempotent matrix does not have zero entry.

LEMMA 1.2. [1, Lemma 1.3] *If* A *is an* $n \times n$ $(n \geqq 2)$ *irreducible sign idempotent matrix, then* A *is entrywise nonzero.*

We note that any sign idempotent matrix with zero entry is reducible.

LEMMA 1.3. *If* $A \in SI$ *has all zero diagonal blocks, then* A *is a zero pattern matrix.*

PROOF. Since A has a zero entry, A is reducible. therefore, A is a zero pattern matrix by *the upper diagonal completion process.*(In [1]). ∎

THEOREM 1.1. *If $\boldsymbol{A} \in SI$ is an irreducible, then $\boldsymbol{A} = \boldsymbol{A}^T$.*

PROOF. Suppose $\boldsymbol{A} = [a_{ij}] \in$ SI and irreducible. By Lemma 1.1 and 1.2, $a_{ii} = +$ for all $i \in I_n$. For any $i,j \in I_n$ such that $i \neq j$,

$$a_{ii} = \sum_{k=1}^{n} a_{ik}a_{ki} = a_{i1}a_{1i} + \ldots + a_{ij}a_{ji} + \ldots + a_{in}a_{ni}.$$

Since $a_{ii} = +$, $a_{ij}a_{ji} = 0$ or $+$. But $\boldsymbol{A} = [a_{ij}]$ contains only nonzero entries by Lemma 1.2. Thus $a_{ij}a_{ji} = +$ for any $i,j \in I_n$ whenever $i \neq j$. Therefore, $a_{ij} = a_{ji}$, and $\boldsymbol{A} = \boldsymbol{A}^T$. ■

THEOREM 1.2. *If $\boldsymbol{A} \in SI$ is an irredusible, then $\boldsymbol{A}$ allows idempotence.*

PROOF. Let A be *the support matrix* of $\boldsymbol{A} = [\alpha_{ij}] \in Q(\boldsymbol{A})$ defined by

$$A = [a_{ij}] \text{ where } a_{ij} = \begin{cases} 1, & \text{if } \alpha_{ij} = + \\ -1, & \text{if } \alpha_{ij} = - \\ 0, & \text{if } \alpha_{ij} = 0. \end{cases}$$

(Note : For an entry a_{ij}, the *support* of a_{ij} is defined similarly)

Suppose $\boldsymbol{A} \in$ SI is irreducible, i.e., $\boldsymbol{A} = \boldsymbol{A}^T$, then $\frac{1}{n}A \in Q(\boldsymbol{A})$. Let $A^2 = [p_{ij}]$ where A is the support matrix of $\boldsymbol{A}$. Since $p_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$, $sgn[p_{ij}] = sgn[a_{ij}]$. So $\frac{1}{n}A = [\frac{1}{n}a_{ij}] \in Q(\boldsymbol{A})$. Let $(\frac{1}{n}A)^2 = [\widetilde{p_{ij}}]$.

$$\begin{aligned} \widetilde{p_{ij}} &= \sum_{k=1}^{n}(\frac{1}{n}a_{ik})(\frac{1}{n}a_{kj}) \\ &= \frac{1}{n^2}\sum_{k=1}^{n} a_{ij} \\ &= \frac{1}{n^2}\cdot n \cdot a_{ij} \\ &= \frac{1}{n}a_{ij}, \text{ for all } i \text{ and } j. \end{aligned}$$

Thus, we have shown that $(\frac{1}{n}A)^2 = \frac{1}{n}A \in Q(\boldsymbol{A})$. ■

EXAMPLE. Let

$$\mathbf{A} = \begin{pmatrix} + & + & - \\ + & + & - \\ - & - & + \end{pmatrix}.$$

Then $\mathbf{A} \in \mathrm{SI}$ and $\mathbf{A}$ is irreducible. Let A be the support matrix of $\mathbf{A} \in Q(\mathbf{A})$. That is,

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \in Q(\mathbf{A}),$$

then

$$\frac{1}{3}A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Also,

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

2. Reducible sign idempotent matrices

Now, let $\mathbf{A} \in \mathrm{SI}$ be reducible. In *a modified Frobenius normal form* [1], suppose $\mathbf{A} = [\mathbf{A}_{ij}]$ be a n-by-n reducible, partial block sign pattern matrix. Since $\mathbf{A}$ is SI if and only if each off-diagonal block $\mathbf{A}_{ij}$ is obtained using *the upper diagonal completion process.(See.* [1])

LEMMA 2.1. [1, Lemma 2.3] *If $\mathbf{A}$ is an $n \times n$ reducible sign pattern matrix such that $\mathbf{A}_{ii}$ and $\mathbf{A}_{jj}$ are entrywise positive blocks. Then $\mathbf{A}$ is sign idempotent only if the sign pattern of $\mathbf{A}_{ij}$ is obtained as follows:*

(1) *$\mathbf{A}_{ij}$ contains only $+$'s or only $-$'s, or*
(2) $\mathbf{A}_{ij} = 0$.

LEMMA 2.2. [1, Lemma 2.3 and 2.4(i)] *Suppose* $\boldsymbol{A}$ *is an* $n \times n$ *reducible sign idempotent matrix where* $\boldsymbol{A}_{ii}$ *is an* $m_i \times m_i$ *entrywise positive matrix and* $\boldsymbol{A}_{jj}$ *is a 0-block. If* $\boldsymbol{A}_{ij}$ *contains a 0-entry or a + or a* $-$, *then* $\boldsymbol{A}_{ij}$ *is an entrywise 0-column or +-column or* $-$*-column matrix(respectively).*

LEMMA 2.3. [1, Lemma 2.4(ii)] *Suppose* $\boldsymbol{A}$ *is an* $n \times n$ *reducible sign idempotent matrix where* $\boldsymbol{A}_{ii}$ *is a 0-block and* $\boldsymbol{A}_{jj}$ *is an* $m_j \times m_j$ *entrywise positive matrix. If* $\boldsymbol{A}_{ij}$ *contains a 0-entry or a + or a* $-$, *then* $\boldsymbol{A}_{ij}$ *is an entrywise 0-row or +-row or* $-$*-row matrix(respectively).*

THEOREM 2.1. *Let* $\boldsymbol{A}_{ii}$ *and* $\boldsymbol{A}_{jj}$ *are entrywise positive block submatrix of* $\boldsymbol{A} \in SI$ *for some* i *and* j *in* I_n. *If* $\boldsymbol{A}$ *allows idempotence, then* $\boldsymbol{A}_{ij}$ *is a 0-block.*

PROOF. Let $\boldsymbol{A}_{ii}$ and $\boldsymbol{A}_{jj}$ are +-blocks(entrywise positive block matrix) in a modified Frobenius normal form and let $\boldsymbol{A}^2 = [P_{ij}]$ where P_{ij} is a block matrix, such that

$$P_{ij} = \boldsymbol{A}_{ii}\boldsymbol{A}_{ij} + \boldsymbol{A}_{ii+1}\boldsymbol{A}_{i+1j} + \cdots + \boldsymbol{A}_{ij}\boldsymbol{A}_{jj} = \boldsymbol{A}_{ij}.$$

By Lemma 2.1, each signs of entries in $\boldsymbol{A}_{ij}$ are same because $\boldsymbol{A}_{ii}$ and $\boldsymbol{A}_{jj}$ are +-blocks. Therefore the sign is determind by these two terms. Thus we only need to consider such $\boldsymbol{A}_{ii}\boldsymbol{A}_{ij}$ and $\boldsymbol{A}_{ij}\boldsymbol{A}_{jj}$.
Therefore, we also know that

$$(\alpha) \qquad \boldsymbol{A}_{ii}\boldsymbol{A}_{ij} + \boldsymbol{A}_{ij}\boldsymbol{A}_{jj} = \boldsymbol{A}_{ij}$$

Now, we multiply each side of (α) by $\boldsymbol{A}_{ii}$, then

$$\boldsymbol{A}_{ii}\boldsymbol{A}_{ii}\boldsymbol{A}_{ij} + \boldsymbol{A}_{ii}\boldsymbol{A}_{ij}\boldsymbol{A}_{jj} = \boldsymbol{A}_{ii}\boldsymbol{A}_{ij}.$$

Thus,

$$(\beta) \qquad \underbrace{\boldsymbol{A}_{ii}\boldsymbol{A}_{ij}}_{(1)} + \underbrace{\boldsymbol{A}_{ii}\boldsymbol{A}_{ij}\boldsymbol{A}_{jj}}_{(2)} = \underbrace{\boldsymbol{A}_{ii}\boldsymbol{A}_{ij}}_{(3)}$$

In the sign pattern matrix, since $\boldsymbol{A}_{ii}$ and $\boldsymbol{A}_{jj}$ are +-blocks, though (1),(2) and (3)terms are not 0, i.e., though $\boldsymbol{A}_{ij}$ is not 0-block, (β) is true.

this is the result of Lemma 2.1. So, $Q(A)$ must contains real matrix that satisfy (α) in order to allow idempotence. But, in fact, for any

$$A_{ii}A_{ij}A_{jj} = A_{ii}A_{ij} - A_{ii}A_{ij} = 0$$

Since A_{ii} and A_{jj} are entrywise positive, $A_{ij} = 0$. Therefore for the case that A_{ii} and A_{jj} are +-block, A_{ij} must be 0-block in order to allow idempotent. ■

From the above Theorem, we know that if $A \in$ SI allows idempotence, then A_{ij} is a 0-block whenever A_{ii} and A_{jj} are +-block. In order to simplify our argument, we only need to consider the cases that satisfies the above fact. Let the set, $\{A \in \text{SI} \mid A_{ij}$ is a 0-block whenever A_{ii} and A_{jj} are +-block $\}$, to be *SIO*.

THEOREM 2.2. *If a reducible $A \in SIO$ has no $0-$block in the main diagonal, then it allows idempotence.*

PROOF. It comes directly from the hypothesis that A is a direct sum of irreducible sign idempotent matrix. ■

THEOREM 2.3. *If a reducible $A \in SIO$ have a $0-$block at the top or bottom, then it allows idempotence.*

PROOF. Let $A \in$ SIO is reducible. If A has a 0-block at the top, without loss of generality (*See* Lemma 2.1), suppose A_{11} is a (1×1)-zero block. Let **A** be a sign pattern matrix as following ;

$$A = \begin{pmatrix} 0 & \vdots & & \delta & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & & & \\ \vdots & \vdots & & \widehat{A} & \\ 0 & \vdots & & & \end{pmatrix}.$$

Now, we consider the following cases in order to show that a real idempotent matrix exists in $Q(A)$;

(i) $\widehat{A} \in$ SI is irreducible.

(ii) $\widehat{\boldsymbol{A}}$ is a direct sum of irreducible sign idempotent matrices.

For the case of (i), suppose $\widehat{\boldsymbol{A}} \in$ SI is irreducible. We may note that $\widehat{\boldsymbol{A}}$ is a positive $(n-1)\times(n-1)$ block sign pattern matrix because of Lemma 1.1 and 1.2. Let $A \in Q(\boldsymbol{A})$ be the support matrix of $\boldsymbol{A}$.
We define a $1\times(n-1)$ matrix $\boldsymbol{\delta} = [a_{12}, a_{13}, \cdots, a_{1n}]$ and

$$\tilde{\boldsymbol{\delta}} = r\cdot[a_{12}, a_{13}, \cdots, a_{1n}] = [\widetilde{a_{12}}, \widetilde{a_{13}}, \cdots, \widetilde{a_{1n}}] \text{ for some } r \in R^{+}.$$

Therefore, $|\widetilde{a_{1i}}| = r$ for all $i, 2 \le i \le n$.

Since $\widehat{A} = [+1]_{(n-1)\times(n-1)}$, we now define an $(n-1)\times(n-1)$ matrix $\widehat{A}$ by

$$\widetilde{A} = \frac{1}{n-1}\cdot\widehat{A} \text{ where } \widehat{A} \text{ is a support matrix of } \widehat{\boldsymbol{A}} \in Q(\widehat{\boldsymbol{A}}).$$

Then

$$\widetilde{A} = [\widetilde{a}_{ij}] = \left(\begin{array}{c:ccc} 0 & r\cdot a_{12} & \cdots & r\cdot a_{in} \\ \hdashline & & & \\ 0 & & J_{n-1} & \\ & & & \end{array}\right) \in Q(\boldsymbol{A})$$

where $J_k = [\frac{1}{k}]_{k\times k}$. Since $\widetilde{A} \in M_{n-1}(R)$ is in the case of irreducible, so we only need to check on p_{1i} for all $i, i \in \{2,\cdots,n\}$. Since $a_{1k}a_{ki}$ has all nonzero signs for each k, it has the same sign as a_{1i}, for each i, $i \in \{2,\cdots,n\}$.

$$\begin{aligned} p_{1i} &= \sum_{k=2}^{n} r\cdot\frac{1}{n-1}\cdot\text{support}(\text{sign } a_{1k}a_{ki}) \\ &= r\cdot\frac{1}{n-1}\sum_{k=2}^{n}\text{support}(\text{sign } a_{1k}a_{k}) \\ &= r\cdot\frac{1}{n-1}\cdot\text{support}(\text{sign } a_{1i}) \\ &= r\cdot\text{support}(\text{sign } a_{1i}) \\ &= \widetilde{a_{1i}}. \end{aligned}$$

Therefore, $\widetilde{A}^2 = \widetilde{A} \in Q(\boldsymbol{A})$.
If $\boldsymbol{A}$ has a 0-block at the bottom, without loss of generality, suppose $\boldsymbol{A}_{nn} = 0$. By using column instead of row, similarly we can show the allowance of idempotent. ■

EXAMPLE. Let

$$\boldsymbol{A} = \begin{pmatrix} 0 & \vdots & - & - & - \\ \cdots & & \cdots & \cdots & \cdots \\ 0 & \vdots & + & + & + \\ 0 & \vdots & + & + & + \\ 0 & \vdots & + & + & + \end{pmatrix} \in SIO.$$

From the above algorithm, we can easily find $\widetilde{\boldsymbol{A}} \in Q(\boldsymbol{A})$ that shows it allows idempotence. We may note that $\widetilde{A}^2 = \widetilde{A}$ and

$$\widetilde{A} = \begin{pmatrix} 0 & -7 & -7 & -7 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

NOTE: For the case that $\boldsymbol{A}_{11} \in M_k$ where $k > 1$, we now use Lemma 2.1. If $\boldsymbol{A}_{11}$ is a 2×2 0-block and

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & \vdots & + & + & + \\ 0 & 0 & \vdots & - & - & - \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & + & + & + \\ 0 & 0 & \vdots & + & + & + \\ 0 & 0 & \vdots & + & + & + \end{pmatrix},$$

similarly we can easily find an idempotent matrix

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & 5 & 5 & 5 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \in Q(\boldsymbol{A}).$$

Since the signature similarity preserves allowance of idempotent, the cases that signs are changed has been taken care.

For the case of (ii), suppose $\widehat{\boldsymbol{A}}$ is a direct sum of irreducible block. Let $A = [a_{ij}]_{n\times n} \in Q(\boldsymbol{A})$ be the support matrix of $\boldsymbol{A}$. Then we define a $1 \times (n-1)$ matrix $\boldsymbol{\delta} = [a_{12}, a_{13}, \cdots, a_{1n}]$ and an $n \times n$ block matrix

$$\widetilde{A} = \begin{pmatrix} 0 & \bar{r}_2 \cdots \bar{r}_2 & \bar{r}_3 \cdots \bar{r}_3 & \cdots & \bar{r}_n \cdots \bar{r}_n \\ & J_{k_2} & 0 & \cdots & 0 \\ & & J_{k_3} & \cdots & 0 \\ & 0 & & \ddots & 0 \\ & & & & J_{k_m} \end{pmatrix}$$

where $\bar{r}_i = r_i$·support(signa_i), for some $r_i \in R^+$. Also, we only need to check $[p_{1i}], 2 \le i \le n$,

$$\begin{aligned} p_{1i} &= \sum_{j=1}^{k_i} r_i \cdot \frac{1}{k_i} \cdot \text{support}(\text{sign} a_{1j} a_{ji}) \\ &= r_i \cdot \frac{1}{k_i} \cdot \text{support}(\text{sign} a_{1i}) \\ &= r_i \cdot \text{support}(\text{sign} a_{1i}) \\ &= \widetilde{a_{1i}}. \end{aligned}$$

Therefore, $\widetilde{A}^2 = \widetilde{A} \in Q(\boldsymbol{A})$

We may note the following, let

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & + & + & - & - \\ 0 & 0 & - & - & + & + \\ 0 & 0 & + & + & 0 & 0 \\ 0 & 0 & + & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & 0 & + & + \end{pmatrix} \in \text{SIO},$$

then, there exists an idempotent matrix $\widetilde{A}$,

$$\widetilde{A} = \begin{pmatrix} 0 & 0 & 5 & 5 & -7 & -7 \\ 0 & 0 & -3 & -3 & 9 & 9 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

THEOREM 2.4. *If a reducible SIO has some 0−block in the main diagonal, then it allows idempotence.*

PROOF. Let $A \in$ SIO is reducible and A_{ii} is a 0-block for some i, where $A = \sum_{i=1}^{m} A_{kk}$. For the case $i = 1$ and m, it was shown thatit allows idempotence in Theorem 2.3. By Theorem 2.1,

$$A_{hj} \text{ must be a 0-block, for any } h, j \in I_n \text{ such that } h, j \neq i,$$

in order to allow idempotent. Therefore, by Lemma 2.1 and 2.2, since each column(resp.,row) entry of $A_{ik}(resp., A_{ki})$ has same signs, the real matrix exists in $Q(A)$,that each column(resp.,row) of $\widetilde{A_{ik}}(resp., \widetilde{A_{ki}})$ have same absolute value. And $[p_{ik}]$ is determined by the case of (i) and (ii) in Theorem 2.3. So, the proof is completed. ■

EXAMPLE. Let

$$A = \begin{pmatrix} + & + & 0 & - & + & 0 & 0 & 0 & 0 \\ + & + & 0 & - & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 & + & + & + & + \end{pmatrix},$$

then there exists an idempotent matrix $\widetilde{A} \in Q(A)$,

$$\widetilde{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & -7 & 5 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -7 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

We note that if A_{ii} and A_{jj} are 0-blocks, since the sign of A_{ij} is determined by $A_{i,j-1}$ and $A_{i+1,j}$ in *the upper diagonal completion process*, real entries are determined by $\widetilde{A}_{i,j-1}$ and $\widetilde{A}_{i+1,j}$ of $\widetilde{A}$. As a conclusion, for any reducible matrix in SIO(*i.e.*, a subset of SI),we have shown that there exist a real idempotent matrix using procedure from proofs in Theorem 2.1 to 2.4 in reverse order. This leads the following Theorem ;

THEOREM 2.5. *If A is a reducible SI ,then A allow idempotence except the case that there is a pair (i,j) such that A_{ij} is a + or − block even though A_{ii} and A_{jj} are + blocks.*

EXAMPLE. Let

$$A = \begin{pmatrix} 0 & - & 0 & + & 0 & 0 & + \\ 0 & + & 0 & - & 0 & 0 & - \\ 0 & 0 & 0 & 0 & - & - & + \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & - \\ 0 & 0 & 0 & 0 & + & + & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in SI,$$

then, there exists an idempotent matrix $\widetilde{A} \in Q(A)$

$$\widetilde{A} = \begin{pmatrix} 0 & -4 & 0 & 28 & 0 & 0 & 32 \\ 0 & 1 & 0 & -7 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 & -3 & -3 & 30 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -5 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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