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THE SCHUR GROUP OF A KRULL DOMAIN

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ABSTRACT. We consider the Schur groups of some module categories, which are subcategories of category of divisorial modules over a Krull domain. Then we obtain the exact sequence connecting class group, Schur class group and Schur groups of these categories.

I. Introduction

A central separable R-algebra A of a commutative ring R represents an equivalence class $[A]$ in $S(R) \subset B(R)$, the Brauer group of R [2], if there is a finite group G and an R-algebra epimorphism $f : RG \longrightarrow A$. This $S(R)$ is called the Schur group of R. If R is a field, a consequence of the Brauer-Witt theorem [8] says that every element of $S(R)$ is represented by a cyclotomic R-algebra.

But for a commutative ring R, the classes in $B(R)$ represented by cyclotomic algebras form a subgroup $S'(R)$ of $S(R)$. Another subgroup $S''(R)$ of $S(R)$ consists of all algebra classes for which the group ring RG is separable.

F. DeMeyer and R. Mollin showed that if R is a commutative ring of positive characteristic then $S(R)$ vanishes [3]. Thus we will assume that R has characteristic zero in this paper. We further assume that R is a Krull domain with quotient field K unless otherwise stated.

In [1] and [2], it was shown that if R is a noetherian integrally closed domain and $[A]$ is in $B(R)$, then $[A]$ is in the kernel of $B(R) \longrightarrow B(K)$ if and only if A is isomorphic to $End_R(M)$ for a reflexive R-module M of finite type. It was also proved that there exist an exact sequence between the class group and the Brauer group of R. In 1981, M. Orzech [7]

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generalized these facts not only for noetherian integrally closed domains but for module categories over Krull domains.

We first review the divisorial R-lattices (i.e reflexive R-module over a Krull domain R) and its basic properties [5,7].

We then discuss the kernel of the homomorphism $S(R) \longrightarrow S(K)$ which is induced by $B(R) \longrightarrow B(K)$. The Schur group for a reflexive R-module category of Krull domain R is defined. We also show that it is possible to obtain the following exact sequence of class groups and Schur groups for R-module categories $\underline{S} \subseteq \underline{M}$ satisfying certain axioms;

$$0 \longrightarrow C\ell(\underline{S}) \longrightarrow C\ell(\underline{M}) \longrightarrow SC\ell(\underline{S},\underline{M}) \longrightarrow S(\underline{S}) \longrightarrow S(\underline{M}).$$

Next, we are concerned with the functoriality for the reflexive R-module categories over a Krull domain.

We will show that if $R[x]$ is the polynomial ring of a Krull domain R then the quotient group $SC\ell(R[x])/SC\ell(R)$ coincides with the kernel of $S(R[x]) \longrightarrow S(R)$.

II. Reflexive modules over Krull domains

Let R be a Krull domain with quotient field K. For the definition and basic properties of Krull domains we refer [5].

An R-module M is said to be *divisorial* if it is torsion-free and in $K \otimes_R M$ if there is the equality

$$M = \bigcap_{p \in Z} M_p.$$

For an R-module M, $rank(M)$ is defined to be $dim_K(K \otimes_R M)$. An R-module M is said to be a *R-lattice* if M is torsion-free of finite rank and there exists a free R-module F of finite type with $M \subseteq F \subseteq K \otimes_R M$. It follows that $rank(M) = rank(F)$ and that $dF \subseteq M$ for a suitable nonzero element d in K. An R-module M is a divisorial R-lattice if and only if the canonical homomorphism $M \to M^{**}$ is an isomorphism, where the M^{**} is the double dual of M , i.e. M is a reflexive R-module.

Let M and N be torsion-free R-modules and MN denote the image of $M \otimes_R N$ in $(K \otimes_R M) \otimes_R (K \otimes_R N)$. Following Yuan [9], we introduce the notion of a modified tensor product.

Now set

$$M \otimes'_R N = \bigcap_{p \in Z} (MN)_p.$$

Then there is a natural map from $M \otimes_R N$ to $M \otimes'_R N$.

PROPOSITION 2.1. [7] *Let M be a flat R-module. Then M is divisorial. Moreover for any divisorial R-lattice N, the natural map $M \otimes_R N \longrightarrow M \otimes'_R N$ is an isomorphism.*

THEOREM 2.2. [7] *Let A be an R-algebra and let M and N be divisorial R-lattices which are also A-modules. Then*
(a) *$Hom_A(M, N)$ is a divisorial R-lattice.*
(b) *There is a natural isomorphism*

$$End_R(M) \otimes'_R End_R(N) \longrightarrow End_R(M \otimes'_R N).$$

We will use the fact that if M is a reflexive R-module then $End_R(M)$ is a central R-algebra. We have a natural map

$$\varphi \, ; \; End_R(M)^\circ \longrightarrow End_R(M^*),$$

sending f° to f^*. The map φ is an isomorphism when localized at each p in Z, so φ is an isomorphism . Let $A = End_R(M)$. Then

$$A \otimes'_R A^\circ \simeq End_R(M \otimes'_R M^*).$$

But $M \otimes'_R M^*$ and $End_R(M)$ are isomorphic by the map sending $x \otimes_R f$ to $f(x)$ in $End_R(M)$. Thus we obtain the isomorphism $A \otimes'_R A^\circ \simeq End_R(A)$.

We record some basic facts about $\otimes'_R$, some of which were noted by Yuan for modules of finite type over noetherian domain R.

THEOREM 2.3. [7, 9] *Let L, M, N and M_i be divisorial R-modules. Then*
(a) *$M \otimes'_R N$ is divisorial.*
(b) *If M and N are R-lattices, so is $M \otimes'_R N$.*
(c) $(L \otimes'_R M) \otimes'_R N = L \otimes'_R (M \otimes'_R N)$
(d) $L \otimes'_R (\oplus M_i) = \oplus (L \otimes'_R M_i)$

(e) $M \otimes'_R N = N \otimes'_R M$

III. The Schur group for module categories

Let R be a Krull domain with quotient field K and let $\underline{D}(R)$ be the category of divisorial R-lattices. We shall consider a subcategory $\underline{M}(R)$ of $\underline{D}(R)$ of divisorial R-lattices satisfying the following axioms ;

(A1) $R \in \underline{M}(R) \subseteq \underline{D}(R)$

(A2) If M and N are in $\underline{M}(R)$, $M \otimes'_R N$ is in $\underline{M}(R)$.

(A3) If M and N are in $\underline{M}$(R), $Hom_R(M, N)$ is in $\underline{M}(R)$.

(A4) If M in $\underline{M}(R), N$ in $\underline{D}(R)$ and $M \otimes'_R N$ is in $\underline{M}(R)$, then N is in $\underline{M}(R)$.

Such a category was considered by M. Orzech [7].

For a category $\underline{M}(R)$ satisfying these axioms, he define the group $B(\underline{M}(R))$ which is analogous to the Brauer group of R in the following way.

For any R-algebra A which is a divisorial R-lattice, there is a natural map

$$\eta_A : A \otimes'_R A^\circ \to End_R(A)$$

which is induced by the usual map from $A \otimes_R A^\circ$ to $End_R(A)$.

Let $A_z(\underline{M}(R))$ be the set of isomorphism classes of central R-algebras which are in $\underline{M}(R)$ as R-modules and for which the natural maps η_A are isomorphisms. Define an equivalence relation $\sim$ on $A_z(\underline{M}(R))$ by $A \sim B$ if $A \otimes'_R End_R(M) \simeq B \otimes'_R End_R(N)$ for some M, N in $\underline{M}(R)$.

Define $B(\underline{M}(R))$ to be the set of equivalence classes of $A_z(\underline{M}(R))$ relative to $\sim$. Let $[A]$ be the equivalence class of A in $B(\underline{M}(R))$. Then $B(\underline{M}(R))$ is an abelian group under the multiplication $[A][B] = [A \otimes'_R B]$ which is well defined by Theorem 2.2 and (A2). Obviously it is commutative, associative by Theorem 2.3, and possesses an identity $[R]$. The inverse of $[A]$ is $[A^\circ]$. We call $B(\underline{M}(R))$ the Brauer group of the category $\underline{M}(R)$.

REMARK 1. Three particular cases of this construction are
(1) $\underline{M}(R) = \underline{D}(R)$ =divisorial R-lattices
(2) $\underline{M}(R) = \underline{P}(R)$ =projective R-modules of finite type
(3) $\underline{M}(R) = \underline{F}(R)$ =free R-modules of finite type

In the case(1), $B(\underline{D}(R))$ is $\beta(R)$, the modified Brauer group considered by Yuan [9] and by B. Auslander in [1]. The group $B(\underline{P}(R))$ of case (2) is the usual Brauer group. The group $B(\underline{F}(R))$ of case (3) is a subgroup of Hoobler's Brauer group, which was considered by Hoobler, Grothendieck and Garfinkel.

Since $B(\underline{D}(R)) \to B(K)$ is one-to-one, the $ker(B(R) \to B(K))$ is the same as $ker(B(\underline{P}(R)) \to B(\underline{D}(R)))$.

To define a subgroup $S(\underline{M}(R))$ of $B(\underline{M}(R))$ which is analogous to the Schur group $S(R)$ of R, we assume that $\underline{M}(R)$ is a category satisfying axioms (A1)-(A4). First, we need the following lemma.

LEMMA 3.1. *Let A and B be central R-algebras which are in $\underline{M}(R)$ as R-modules. If there are finite groups G, H and R-algebra epimorphisms $f : RG \to A$, $g : RH \to B$, then the correspondence $(x, y) \longmapsto f(x) \otimes'_R g(y)$ induces an R-algebra epimorphism $R(G \times H) \to A \otimes'_R B$.*

PROOF. We know that there exist an epimorphism from $R(G \times H)$ to $A \otimes_R B$ defined by $(x, y) \to f(x) \otimes_R g(y)$.

For each height one prime $p \in Z$, we have an epimorphism

$$(R(G \times H))_p \to (A \otimes_R B)_p. \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

We also have isomorphisms

$$(A \otimes_R B)_p \simeq A_p \otimes_R B_p \simeq A_p \otimes'_R B_p \simeq (A \otimes'_R B)_p \quad \cdots\cdots \quad (2)$$

for all height one prime $p \in Z$, by Proposition 2.1.

Since the free R-module $R(G \times H)$ is divisorial R-lattice and $A \otimes'_R B$ is divisorial R-lattice by Theorem 2.3, the correspondence $(x, y) \to f(x) \otimes'_R g(y)$ induces an R-algebra epimorphism $R(G \times H) \to A \otimes'_R B$ by (1) and (2).

Let $S(\underline{M}(R))$ be the set of central R-algebra classes in $B(\underline{M}(R))$ which are epimorphic images of the finite group rings over R. Then, it is seen to see the following theorem holds;

THEOREM 3.2. *$S(\underline{M}(R))$ is a subgroup of $B(\underline{M}(R))$.*

We call $S(\underline{M}(R))$ the Schur group of the category $\underline{M}(R)$ over Krull domain R.

REMARK 2. If $\underline{M}(R)$ is the category of projective R-modules of finite type, $S(\underline{M}(R))$ is the usual Schur group of R.

For a Krull domain R, let $\underline{M}(R)$ be a category of R-modules satisfying axioms (A1)-(A4). Let we recall the following from [2] and [4] ; $Cl(\underline{M}(R))$ to be the set of isomorphism classes $\{I\}$ of R-modules I of rank 1 which are in $\underline{M}(R)$. It is easy to see that $Cl(\underline{M}(R))$ is closed under the operation $\otimes_R'$. Furthermore, $Cl(\underline{M}(R))$ is a group under this operation, the inverse of $\{I\}$ being given by $\{I^*\}$. This can be easily seen by noting that $I \otimes_R' I^* \to End_R(I)$ is an isomorphism, and that $End_R(I) \simeq R$.

To define a group $BCl(\underline{S}(R), \underline{M}(R))$ for categories $\underline{S}(R)$ and $\underline{M}(R)$ satisfying axioms (A1)-(A4) and $\underline{S}(R) \subseteq \underline{M}(R)$, consider the set of isomorphism classes $\underline{C}$ of objects M in $\underline{M}(R)$ for which $End_R(M)$ is in $\underline{S}(R)$. Define a relation $\sim$ on $\underline{C}$ by $M \sim N$ if $M \otimes_R' P \simeq N \otimes_R' Q$ for some P and Q in $\underline{S}(R)$. Then the relation $\sim$ is clearly an equivalence relation and $BCl(\underline{S}(R), \underline{M}(R)) = \underline{C} \,/ \sim$ is an abelian group with multiplication induced by $\otimes_R'$. The inverse of the class $[M]$ of M is $[M^*]$, since $M \otimes_R' M^* \simeq End_R(M)$.

In [7], M.Orzech showed that there is an exact sequence

$$\begin{aligned} 1 \to Cl(\underline{S}(R)) \to Cl(\underline{M}(R)) \to BCl(\underline{S}(R), \underline{M}(R)) \\ \to B(\underline{S}(R)) \to B(\underline{M}(R)) \end{aligned}$$

where $\underline{S}(R)$ and $\underline{M}(R)$ are reflexive R-module categories satisfying axioms (A1)-(A4).

We are going to relate the class group and the Schur group with $BCl(\underline{S}(R), \underline{M}(R))$. For this, we consider subset $SCl(\underline{S}(R), \underline{M}(R))$ of

$BCl(\underline{S}(R), \underline{M}(R))$ consisting of algebra classes $[M]$ in $BCl(\underline{S}(R), \underline{M}(R))$ such that there exists a finite group G and an epimorphism $RG \to End_R(M)$. Then $SCl(\underline{S}(R), \underline{M}(R))$ is a subgroup of $BCl(\underline{S}(R), \underline{M}(R))$. We can show that there exist an exact sequence between these groups as following.

THEOREM 3.3. *Let R be a Krull domain. Assume $\underline{S}(R)$ and $\underline{M}(R)$ are two categories of divisorial R-lattices satisfying axioms (A1)-(A4) and $\underline{S}(R) \subseteq \underline{M}(R)$, then there is an exact sequence*

$$1 \to Cl(\underline{S}(R)) \xrightarrow{i} Cl(\underline{M}(R)) \xrightarrow{j} SCl(\underline{S}(R), \underline{M}(R)) \xrightarrow{\alpha} S(\underline{S}(R)) \xrightarrow{\beta} S(\underline{M}(R))$$

where $i\{A\} = \{A\}, j\{M\} = [M], \alpha[M] = [End_R(M)], \beta[A] = [A]$.

PROOF. The maps are well-defined as in [7]. This is clear for i and β. For j, if $\{M\} \in Cl(\underline{M}(R))$, $End_R(M) \simeq M \otimes_R' M^* \simeq R$. Thus $j(Cl(\underline{M}(R))) \subset SCl(\underline{S}(R), \underline{M}(R))$. To verify it for α, let $[M]$ be in $SCl(\underline{S}(R), \underline{M}(R))$. By definition of our $SCl(\underline{S}(R), \underline{M}(R))$, $End_R(M) \in \underline{S}(R)$ and $[End_R(M)]$ is in $S(\underline{S}(R))$. If $[M] = [N]$ in $SCl(\underline{S}(R), \underline{M}(R))$, the definition of the relation $\sim$ in $\underline{C}$ implies that $\alpha[M] = \alpha[N]$. And for $[M]$ contained in $Cl(\underline{M}(R))$, since M is a reflexive R-module of rank 1 and $M \otimes_R' M^* \simeq R$. Thus $\alpha j = 1$. The exactness of the sequence at the other places is shown by M. Orzech [7].

REMARK 3. Let $SCl(R)$ be $SCl(\underline{P}(R), \underline{D}(R))$. Then by Remark 1, $ker(S(R) \to S(K)) = SCl(R)$. If $S(R) \to S(K)$ is 1-1, $S''(R)$ is a subgroup of $S'(R)$ as in proposition 2[3]. However the proof of the proposition is not valid for noetherian domain such that $B(R) \to B(K)$ is not 1-1. For $R = \frac{\Re[x,y,z]}{(x^2+y^2+z^2)}$, $(\frac{-1,-1}{R})$ is non-trivial element in $ker(B(R) \to B(K))$, and $(\frac{-1,-1}{R}) \in SCl(R)$.

It would be interesting to know whether $SCl(R) = BCl(R)$. It is not clear whether 2 torsion subgroups of $SCl(R)$ and $BCl(R)$ coincide.

IV. Functoriality

Let R and T be Krull domains with $R \subseteq T$. The inclusion of R in T is said to be a *Krull morphism* if for every height one prime q of T, $q \cap R$ has height at most one; this condition is called (NBU), which means " no blowing up".

This condition is useful because when it holds there is an induced map from the class group of R to that of T. Let $\mathcal{P}$ be a height one prime of T and $p = \mathcal{P} \cap R$ be a prime ideal of R. Since $T_{\mathcal{P}}$ is an R_p-algebra and R_p is a discrete valuation ring provided condition (NBU) is satisfied, we see that $T_{\mathcal{P}}$ is a faithfully flat R_p-module. The converse holds as well. That is, if $T_{\mathcal{P}}$ is faithfully flat R_p-module, then dim $R_p \leq$ dim $T_{\mathcal{P}}$ and so height $p \leq 1$ [5].

There are several general cases in which there is no blowing up (NBU) for divisorial ideals.

Examples

In any of the cases below, the R-algebra T satisfies condition (NBU).

(1) The R-algebra T is flat as an R-module.

(2) The R-algebra T is integral over R.

(3) The R-algebra T is a subintersection, namely $T = \bigcap_{p \in Y} R_p$ where Y is a subset of Z.

THEOREM 4.1. [7] *Let $R \subseteq T$ be an inclusion of Krull domains. Then $R \hookrightarrow T$ is a Krull morphism if and only if T is divisorial as an R-module.*

Let $\underline{M}(R)$ be a subcategory of $\underline{D}(R)$ satisfying (A1) - (A3) of section III, and let the inclusion i : $R \hookrightarrow T$ be a Krull morphism, namely T is divisorial as an R-module. Then the correspondence $M \longmapsto T \otimes'_R M$ induces a funtor $i^* : \underline{M}(R) \longrightarrow \underline{M}(T)$.

In [7], M. Orzech showed that there is a group homomorphism from $B(\underline{M}(R))$ ($C\ell(\underline{M}(R))$ and $BC\ell(\underline{S}(R), \underline{M}(R))$, resp.) to $B(\underline{M}(T))$ ($C\ell(\underline{M}(T))$ and $BC\ell(\underline{S}(T), \underline{M}(T))$, resp.) with the following commutative diagram

(4.1)

$$\begin{array}{ccccccccccc} 1\to & \mathcal{C}\ell(\underline{S}(R)) & \to & \mathcal{C}\ell(\underline{M}(R)) & \to & B\mathcal{C}\ell(\underline{S}(R),\underline{M}(R)) & \to & B(\underline{S}(R)) & \to & B(\underline{M}(R)) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1\to & \mathcal{C}\ell(\underline{S}(T)) & \to & \mathcal{C}\ell(\underline{M}(T)) & \to & B\mathcal{C}\ell(\underline{S}(T),\underline{M}(T)) & \to & B(\underline{S}(T)) & \to & B(\underline{M}(T)). \end{array}$$

Also we define a group homomorphism from $S(\underline{M}(R))$ (resp. $S\mathcal{C}\ell(\underline{S}(R), \underline{M}(R))$) to $S(\underline{M}(T))$ (resp. $S\mathcal{C}\ell(\underline{S}(T),\underline{M}(T))$) and have a commutative diagram between these groups over divisorial R-module categories.

LEMMA 4.2. *If $\underline{M}(R)$ is a subcategory of $\underline{D}(R)$ satisfying axioms (A1)-(A3) and $i : R \hookrightarrow T$ is a divisorial morphisn of Krull domains R and T with $R \subseteq T$, then the correspondence $[A] \longmapsto [T \otimes'_R A]$ induces a group homormophism $S(i) : S(\underline{M}(R)) \longrightarrow S(\underline{M}(T))$.*

And for another divisorial morphism $j : T \hookrightarrow U$ of Krull domains, we have

$$S(ji) = S(j)S(i).$$

If axiom A(4) holds as well, then there is a group homomorphism

$$S\mathcal{C}\ell(i) : S\mathcal{C}\ell(\underline{S}(R),\underline{M}(R)) \longrightarrow S\mathcal{C}\ell(\underline{S}(T),\underline{M}(T))$$

which sends $[M]$ to $[T \otimes'_R M]$.

PROOF. For [A] in $S(\underline{M}(R))$, there is a finite group G and an epimorphism $RG \longrightarrow A$. And this induces an epimorphism $TG \longrightarrow A \otimes_R T$. Hence $T_{\mathcal{P}}G \longrightarrow (A \otimes_R T)_{\mathcal{P}}$ is onto for all height one prime ideals $\mathcal{P}$ in T.

$$(A \otimes_R T)_{\mathcal{P}} \simeq A_p \otimes_{R_p} T_{\mathcal{P}} \simeq (A \otimes_R T)_p.$$

Since $(A \otimes_R T)_p \simeq (A \otimes'_R T)_{\mathcal{P}}$, the induced map $TG \longrightarrow A \otimes'_R T$ is an epimorphism. Also the property $B(ji) = B(j)B(i)$ implies $S(ji) = S(j)S(i)$.

Thus we have the following theorem.

THEOREM 4.3. *Let $\underline{S}(R)$ and $\underline{M}(R)$ be subcategories of $\underline{D}(R)$ satisfying axioms (A1) - (A4) and $\underline{S}(R) \subseteq \underline{M}(R)$. Let $i : R \longrightarrow T$ be a divisorial morphism of Krull domains. Then there is a commutative diagram with exact rows ;*

(4.2)

$$\begin{array}{ccccccccccc} 1 \to & C\ell(\underline{S}(R)) & \to & C\ell(\underline{M}(R)) & \to & SC\ell(\underline{S}(R),\underline{M}(R)) & \to & S(\underline{S}(R)) & \to & S(\underline{M}(R)) \\ & \downarrow i_1 & & \downarrow i_2 & & \downarrow i_3 & & \downarrow i_4 & & \downarrow i_5 \\ 1 \to & C\ell(\underline{S}(T)) & \to & C\ell(\underline{M}(T)) & \to & SC\ell(\underline{S}(T),\underline{M}(T)) & \to & S(\underline{S}(T)) & \to & S(\underline{M}(T)) \end{array}$$

where $i_j\{M\} = \{T \otimes'_R M\}$, $j = 1, 2$ *and* $i_k[M] = [T \otimes'_R M]$, $k = 3.4.5$.

V. The Schur group of polynomial rings over a Krull domain

Let R be a Krull domain with quotient field K which is perfect and $R[x]$ be the ring of polynomials over R. Since $R[x]$ is flat as an R-module, we have the following commutative diagrams with exact rows. We assume that R-module categories $\underline{S}(R)$ and $\underline{M}(R)$ satisfy the same conditions as in the previous sections.

From the diagrams of (4.1) and (4.2) in section IV, we have

(5.1)

$$\begin{array}{ccccccccccc} 1 \to & C\ell(\underline{S}(R)) & \to & C\ell(\underline{M}(R)) & \to & BC\ell(\underline{S}(R),\underline{M}(R)) & \to & B(\underline{S}(R)) & \to & B(\underline{M}(R)) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 \to & C\ell(\underline{S}(R[x])) & \to & C\ell(\underline{M}R[x]) & \to & BC\ell(\underline{S}(R[x]),\underline{M}(R[x])) & \to & B(\underline{S}(R[x])) & \to & B(\underline{M}(R[x])) \end{array}$$

(5.2)

$$\begin{array}{ccccccccccc} 1 \to & C\ell(\underline{S}(R)) & \to & C\ell(\underline{M}(R)) & \to & SC\ell(\underline{S}(R),\underline{M}(R)) & \to & S(\underline{S}(R)) & \to & S(\underline{M}(R)) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 \to & C\ell(\underline{S}(R[x])) & \to & C\ell(\underline{M}R[x]) & \to & SC\ell(\underline{S}(R[x]),\underline{M}(R[x])) & \to & S(\underline{S}(R[x])) & \to & S(\underline{M}(R[x])) \end{array}$$

Let $\underline{S}(R)$ be the category of projective R-modules and $\underline{M}(R)$ be the category of divisorial R-lattices. As before $B(\underline{D}(R)) \longrightarrow B(K)$ is one to one, $ker(B(R) \longrightarrow B(K))$ is the same as $ker(B(R) \longrightarrow \Gamma(\underline{D}(R)))$. We also note that $B(K) \simeq B(K[x])$ since K is perfect and that $B(K[x]) \hookrightarrow B(K(x))$ is an inclusion, since $K[x]$ is regular. Hence from (5.1) and (5.2), we have the following diagrams.

(5.1')

$$\begin{array}{ccccccccc}
1 & & 1 & & 1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Pic(R) & \rightarrow & C\ell(R) & \rightarrow & BC\ell(R) & \rightarrow & B(R) & \rightarrow & B(K) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Pic(R[x]) & \rightarrow & C\ell(R[x]) & \rightarrow & BC\ell(R[x]) & \rightarrow & B(R[x]) & \rightarrow & B(K[x]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & BC\ell(R[x])/BC\ell(R) & \rightarrow & B(R[x])/B(R) & & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

(5.2')

$$\begin{array}{ccccccccc}
1 & & 1 & & 1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Pic(R) & \rightarrow & C\ell(R) & \rightarrow & SC\ell(R) & \rightarrow & S(R) & \rightarrow & S(K) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Pic(R[x]) & \rightarrow & C\ell(R[x]) & \rightarrow & SC\ell(R[x]) & \rightarrow & S(R[x]) & \rightarrow & S(K[x]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & SC\ell(R[x])/SC\ell(R) & \rightarrow & S(R[x])/S(R) & & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

The first two vertical sequences are exact and the last three vertical sequences are split exact induced from the canonical maps

$$R \hookrightarrow R[x] \longrightarrow R.$$

Hence we have the following theorem.

THEOREM 5.1. *Let R be a Krull domain with quotient field K. Then*

(1) $BC\ell(R[x])/BC\ell(R) \simeq ker(B(R[x]) \longrightarrow B(R))$ [4]

(2) $SC\ell(R[x])/SC\ell(R) \simeq ker(S(R[x]) \longrightarrow S(R))$

In particular, if R is a regular domain, so is $R[x]$ and $BC\ell(R[x]) = BC\ell(R) = 0$. Hence $B(R[x]) \simeq B(R)$ and $S(R[x]) \simeq S(R)$. We consider some examples.

Examples

(1) For the ring of integers Z, since $B(Z) = 0$, $B(Z[x_1, ..., x_n]) = 0$ and $S(Z[x_1, ..., x_n]) = 0$ with indeterminates $x_1, .., x_n$.

(2) Also we know that $B(R) = B(Z(1/2)) = S(Z(1/2)) \simeq Z/2Z$. For the nontrivial class in $S(Z(1/2))$ has a representative which is a homomorphic image of the group ring of the quaternion group of order 8. And thus $B(Z(1/2)[x_1, ..., x_n]) = S(Z(1/2)[x_1, .., x_n]) \simeq Z/2Z$.

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