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C(S) EXTENSIONS OF S-I-BCK-ALGEBRAS

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ABSTRACT. In this paper we consider more systematically the centralizer C(S) of the set $S = \{f_a \mid f_a : X \to X; x \mapsto x * a, a \in X\}$ with respect to the semigroup End(X) of all endomorphisms of an implicative BCK-algebra X with the condition (S). We obtain a series of interesting results. The main results are stated as follows:

(1) C(S) with respect to a binary operation $*$ defined in a certain way forms a bounded implicative BCK-algebra with the condition (S).
(2) X can be imbedded in C(S) such that X is an ideal of C(S).
(3) If X is not bounded, it can be imbedded in a bounded subalgebra T of C(S) such that X is a maximal ideal of T.
(4) If $X\,(\neq \{0\})$ is semisimple, C(S) is BCK-isomorphic to $\prod_{i\in I} A_i$ in which $\{A_i\}_{i\in I}$ is simple ideal family of X.

0. Introduction

Throughout the paper, the S-P-I-BCK-algebra denote the positive implicative BCK-algebra with the condition (S). Similarly, we have the symbols: S-BCK-algebra, S-I-BCK-algebra, P-I-BCK-algebra, etc.

K. Iséki in [5] made "the Iséki extension" for a BCK-algebra X (see [9]) and he showed that the algebra X^* after this extension is a bounded BCK-algebra with X a maximal ideal of X^*. The present authors in [3] made "the extension of order dual" for a S-I-BCK-algebra X and we obtained that the algebra X^* after this extension is a bounded S-I-BCK-algebra with X a maximal ideal of X^*.

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For a P-I-BCK-algebra X, using the semigroup End(X) of all endomorphisms of X, the present authors in [1] extended X as a S-P-I-BCK-algebra X^* such that X is a subalgebra of X^*. However the X^* might not be bounded. For the case that X^* is non-bounded, using once more the End(X), can we make further extension such that the algebra after that extension not only satisfies the condition (S) but also is bounded? For this question, when X is implicative, this paper will give a definite answer. We call it the C(S) extension.

1. Preliminaries

We shall freely use the concepts and basic properties for S-BCK-algebra, P-I-BCK-algebra and I-BCK-algebra mentioned in [6] and [8].

PROPOSITION 1.1. *Let X be a S-P-I-BCK-algebra and f a BCK-endomorphism of X. Then*

(1) ([7], Theorem 2) $(x \circ y) * z = (x * z) \circ (y * z)$;
(2) ([6], Theorem 16) $f(x \circ y) = f(x) \circ f(y)$.

PROPOSITION 1.2. *Let X be a S-I-BCK-algebra. Then*
(3) $(x * y) \wedge y = 0$;
(4) $x * y = x * (x \wedge y)$;
(5) $x * (y * z) = ((x * y) * z) \circ (x \wedge z)$;
(6) $x = (x * y) \circ (x \wedge y)$;
(7) $(x \wedge y) * z = x \wedge (y * z)$.

PROOF. (3) $(x * y) \wedge y = y * (y * (x * y)) = y * y = 0$.
(4) $x * y = x * (x * (x * y)) = x * (x \wedge y)$.
(5) $(x * (y * z)) * (((x * y) * z) \circ (x \wedge z))$
$= ((x * (y * z)) * ((x * y) * z)) * (x \wedge z)$
$= ((x * (y * z)) * ((x * z) * (y * z))) * (x \wedge z)$
$= ((x * (x * z)) * (y * z)) * (x \wedge z)$ [by X positive implicative]
$= ((z \wedge x) * (y * z)) * (x \wedge z) = 0$.
Hence $x * (y * z) \leq ((x * y) * z) \circ (x \wedge z)$. On the other hand,
$(((x * y) * z) \circ (x \wedge z)) * (x * (y * z))$
$= (((x * y) * z) * (x * (y * z))) \circ ((x \wedge z) * (x * (y * z)))$ [by (1)]

$= (((x * z) * (y * z)) * (x * (y * z))) \circ ((z * (z * x)) * (x * (y * z)))$
$= (((x * z) * x) * (y * z)) \circ ((z * (z * x)) * (x * (y * z)))$
$= (0 * (y * z)) \circ ((z * (x * (y * z)))$
$\quad *((z * (x * (y * z))) * (x * (x * (y * z)))))$
$= 0 \circ ((x * (x * (y * z))) \wedge (z * (x * (y * z))))$
$= (y * z) \wedge x \wedge (z * (x * (y * z)))$
$\leq (y * z) \wedge z = 0.$ [by (3)]
Hence $((x * y) * z) \circ (x \wedge z) \leq x * (y * z)$. It follows that (5) holds.
(6) By (5), $x = x * (y * y) = ((x * y) * y) \circ (x \wedge y) = (x * y) \circ (x \wedge y)$.
(7) $(x \wedge y) * z = (y \wedge x) * z = (x * (x * y)) * z$
$= (x * z) * ((x * y) * z)$ [by X positive implicative]
$= (x * (x \wedge z)) * ((x * y) * z)$ [by (4)]
$= x * ((x \wedge z) \circ ((x * y) * z))$
$= x * (x * (y * z))$ [by (5)]
$= (y * z) \wedge x = x \wedge (y * z)$. □

DEFINITION 1.3 ([7]). Let X be a S-BCK-algebra. A nonempty subset A of X is called an additive ideal of X if (i) $x \in A$ and $y \leq x$ imply $y \in A$ and (ii) $x, y \in A$ imply $x \circ y \in A$.

PROPOSITION 1.4. ([10], Theorem 9.3) *Let X be a S-BCK-algebra. Then A is an ideal of X if and only if A is an additive ideal of X.*

PROPOSITION 1.5. ([4]) *Let X be a S-P-I-BCK-algebra and A_1, A_2 ideals of X. Then the generated ideal $\langle A_1 \cup A_2 \rangle = \{a_1 \circ a_2 \mid a_i \in A_i, i = 1, 2\}$.*

REMARK. We call the above ideal $A = \langle A_1 \cup A_2 \rangle$ as the sum of A_1 and A_2, and denote $A = A_1 \circ A_2$. If $A_1 \cap A_2 = \{0\}$, we call A as the direct sum of A_1 and A_2, and denote $A = A_1 \oplus A_2$ in which every A_i is called a summand of A. The sum or direct sum can be generalized to the case of any many ideals (see [2]).

PROPOSITION 1.6. ([2]) *Let X be a S-P-I-BCK-algebra. Then*
(8) *Every ideal of X is a semigroup with respect to the operation $\circ$;*
(9) *$A \circ (B \circ C) = (A \circ B) \circ C$ where A, B, C are ideals of X.*

It is obvious.

PROPOSITION 1.7. *Let X be a S-P-I-BCK-algebra and $X = A \oplus B$. Then for any $x \in X$, there is an unique representation:*

$$x = a \circ b, \ a \in A, \ b \in B.$$

LEMMA 1.8. *Let X be a S-BCK-algebra and A, B two ideals of X. If $B = B_1 \oplus B_2$ then $A \cap B = (A \cap B_1) \oplus (A \cap B_2)$.*

PROOF. Let $x \in (A \cap B_1) \oplus (A \cap B_2)$ and $x = x_1 \circ x_2$, $x_i \in A \cap B_i, i = 1, 2$. Since the ideal $A \cap B$ is an additive ideal, by $x_i \in A \cap B_i \subseteq A \cap B$, $x = x_1 \circ x_2 \in A \cap B$. On the other hand, put $x \in A \cap B$. By $x \in B = B_1 \oplus B_2$, there is $x = x_1 \circ x_2$, $x_i \in B_i$. By $x \in A$, $x_i \in A$. Then $x_i \in A \cap B_i$ and $x = x_1 \circ x_2 \in (A \cap B_1) \circ (A \cap B_2)$. Also $(A \cap B_1) \cap (A \cap B_2) \subseteq B_1 \cap B_2 = \{0\}$. Hence $A \cap B = (A \cap B_1) \oplus (A \cap B_2)$. □

2. On centralizer C(S) of End(X)

Let X be a S-P-I-BCK-algebra and End(X) the set of all endomorphisms of X. Then End(X) with respect to the composition "·" of maps forms a semigroup with the identity map 1_X. Put

$$S = \{f_a : X \to X \mid f_a(x) = x * a, \, a \in X\}.$$

Note that for all $x, y \in X$,

$$f_a(x * y) = (x * y) * a = (x * a) * (y * a) = f_a(x) * f_a(y).$$

Hence f_a is an endomorphism of X. Since

$$\begin{aligned}(f_a \cdot f_b)(x) &= f_a(f_b(x)) = f_a(x * b) = (x * b) * a \\ &= x * (b \circ a) = x * (a \circ b) = f_{a \circ b}(x)\end{aligned}$$

for all $x \in X$, we obtain $f_a f_b = f_{a \circ b} \in S$. Also $f_a f_b = f_{a \circ b} = f_{b \circ a} = f_b f_a$ and $f_0 f_a = f_{0 \circ a} = f_a$ where 0 is the zero element of X. Thus S is a commutative sub-semigroup of End(X) with the identity map $1_X = f_0$.

We denote the centralizer of S with respect to End(X) by C(S), that is,

$$\mathrm{C}(S) = \{f \in \mathrm{End}(X) \mid f f_a = f_a f \text{ for all } f_a \in S\}.$$

It is obvious that C(S) is also a sub-semigroup of End(X) and $S \subseteq$ C(S).

PROPOSITION 2.1. *Let X be a S-P-I-BCK-algebra and $f \in End(X)$. Then $f \in C(S)$ if and only if $f(x) * a = f(x * a)$ for any $a \in X$.*

PROOF. If $f \in C(S)$, then

$$f(x) * a = f_a(f(x)) = (f_a f)(x) = (f f_a)(x) = f(f_a(x)) = f(x * a).$$

Conversely if $f(x) * a = f(x * a)$ for any $a \in X$, then

$$(f_a f)(x) = f(x) * a = f(x * a) = (f f_a)(x).$$

Hence $f_a f = f f_a$ and so $f \in C(S)$. □

LEMMA 2.2. *Let X be a S-I-BCK-algebra and $f \in End(X)$. Then $f \in C(S)$ if and only if $f(a) \leq a$ for any $a \in X$.*

PROOF. If $f \in C(S)$, then $f(a) * a = f(a * a) = 0$ by Proposition 2.1 and hence $f(a) \leq a$. Conversely if $f(a) \leq a$, then $f(x) * a \leq f(x) * f(a)$ for any $x \in X$. On the other hand,

$$\begin{aligned} &(f(x) * f(a)) * (f(x) * a) \\ =&(((f(x) * f(a)) * f(x)) * a) \circ ((f(x) * f(a)) \wedge a) \qquad [\text{ by } (5)\ ] \\ =&(f(x) * f(a)) \wedge a = f(x * a) \wedge a \\ \leq&(x * a) \wedge a = 0. \qquad [\text{ by } (3)\ ] \end{aligned}$$

Hence $f(x)*a = f(x)*f(a) = f(x*a)$. By Proposition 2.1, $f \in C(S)$. □

PROPOSITION 2.3. *Let X be a S-I-BCK-algebra and $f \in End(X)$. Then the following conditions are equivalent:*

(a) *$f \in C(S)$;*

(b) *Every initial section $A(a) = \{x \in X \mid x \leq a\}$ is invariant under f $\Big(i.e., f(A(a)) \subseteq A(a)\Big)$;*

(c) *$f\big(Im(f_a)\big) = f_a\big(Im(f)\big)$ for all $a \in X$.*

PROOF. (a) $\Longleftrightarrow$ (b) This is a direct consequence of Lemma 2.2.
(a) $\Longleftrightarrow$ (c) This is an immediate conclusion of Proposition 2.1. □

PROPOSITION 2.4. *Let X be a S-P-I-BCK-algebra and $f \in C(S)$. Then $Ker(f) = \{x * f(x) \mid x \in X\}$.*

PROOF. If $x \in \mathrm{Ker}(f)$, then $x = x * 0 = x * f(x)$. On the other hand, for any $x \in X$, $f(x * f(x)) = f(f_{f(x)}(x)) = (f f_{f(x)})(x) = f_{f(x)}(f(x)) = f(x) * f(x) = 0$, which means $x * f(x) \in \mathrm{Ker}(f)$. This completes the proof. □

PROPOSITION 2.5. *Let X be a S-I-BCK-algebra and $f \in C(S)$. Then $f = f^2$.*

PROOF.
$$\begin{aligned} f(x) &= f((x * f(x)) \circ (x \wedge f(x))) && [\text{ by (6) }] \\ &= f((x * f(x)) \circ f(x)) && [\text{ by Lemma 2.2 }] \\ &= f(x * f(x)) \circ f^2(x) && [\text{ by (2) }] \\ &= f^2(x) && [\text{ by Proposition 2.4 }] \end{aligned}$$
and so $f^2 = f$. □

COROLLARY 2.6. *Let X be a S-I-BCK-algebra and $f \in C(S)$. Then $Im(f) = \{x \in X \mid f(x) = x\}$.*

PROOF. If $x \in \mathrm{Im}(f)$, then there exists $y \in X$ such that $f(y) = x$. By Proposition 2.5, $x = f(y) = f^2(y) = f(f(y)) = f(x)$. Thus $\mathrm{Im}(f) \subseteq \{x \in X \mid f(x) = x\}$. It is clear that the inverse containing relation holds. □

THEOREM 2.7. *Let X be a S-I-BCK-algebra and $f \in C(S)$. Then*
(a) $Im(f)$ is an ideal of X;
(b) $X = Ker(f) \oplus Im(f)$.

PROOF. (a) Clearly $0 \in \mathrm{Im}(f)$. If $x, y * x \in \mathrm{Im}(f)$, then
$$\begin{aligned} y &= (y * x) \circ (y \wedge x) && [\text{ by (6) }] \\ &= f(y * x) \circ (y \wedge f(x)) && [\text{ by Corollary 2.6 }] \\ &= f(y * x) \circ (f(x) * (f(x) * y)) \\ &= f(y * x) \circ (f(x) * f(x * y)) && [\text{ by Proposition 2.1 }] \\ &= f(y * x) \circ f(x * (x * y)) \\ &= f((y * x) \circ (y \wedge x)) = f(y). && [\text{ by (6) }] \end{aligned}$$

So $y \in \mathrm{Im}(f)$. It follows that $\mathrm{Im}(f)$ is an ideal of X.

(b) By Lemma 2.2, for any $x \in X$, $x = (x * f(x)) \circ (x \wedge f(x)) = (x * f(x)) \circ f(x)$ in which $x * f(x) \in \mathrm{Ker}(f)$ and $f(x) \in \mathrm{Im}(f)$. Hence $X = \mathrm{Ker}(f) \circ \mathrm{Im}(f)$. Next suppose that $x \in \mathrm{Ker}(f) \cap \mathrm{Im}(f)$. Then $x = f(x)$ by $x \in \mathrm{Im}(f)$ and $f(x) = 0$ by $x \in \mathrm{Ker}(f)$ and so $x = f(x) = 0$. Hence (b) holds. □

PROPOSITION 2.8. *Let X be a S-I-BCK-algebra and $f \in End(X)$. Then the following are equivalent:*

(a) *$f \in C(S)$;*

(b) *$f^2 = f$ and $Im(f)$ is an ideal of X.*

PROOF. (a) $\Longrightarrow$ (b) This is got by Proposition 2.5 and Theorem 2.7.

(b) $\Longrightarrow$ (a) Since $f^2 = f$, with the same as the proof of Corollary 2.6, we have $\mathrm{Im}(f) = \{x \in X \mid f(x) = x\}$. Also note that $(f(x) * x) * f(x) = 0$, by $\mathrm{Im}(f)$ an ideal of X, we get $f(x) * x \in \mathrm{Im}(f)$ for all $x \in X$. Then

$$f(x) * x = f(f(x) * x) = f^2(x) * f(x) = f(x) * f(x) = 0,$$

that is, $f(x) \leq x$. Hence Lemma 2.2 implies $f \in \mathrm{C}(S)$. □

REMARK. In (b) of Proposition 2.8, the condition that $\mathrm{Im}(f)$ is an ideal of X is very strong. Even it becomes that $f^2 = f$ and $\mathrm{Ker}(f)$ is a summand of X, $f \in \mathrm{C}(S)$ might not holds, in fact, we have the following.

EXAMPLE 2.9. It is easy to verify that the set $X = \{0, a, b, 1\}$ with the operation $*$ defined by

$*$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

forms a S-I-BCK-algebra. Put

$$f: X \to X;\ 0 \mapsto 0,\ a \mapsto 0,\ b \mapsto 1,\ 1 \mapsto 1.$$

Then evidently $f^2 = f$. By immediate verification, we obtain that $f \in \mathrm{End}(X)$ and $\mathrm{Ker}(f) = \{0, a\}$ is a summand of X but $f \notin \mathrm{C}(S)$ since $f(b) = 1 \not\leq b$.

THEOREM 2.10. *Let X be a S-I-BCK-algebra. Then C(S) is a commutative sub-semigroup of End(X).*

PROOF. We have already known that $C(S)$ is a sub-semigroup of $\mathrm{End}(X)$. Suppose that $f_1, f_2 \in C(S)$. For any $x \in X$, by Theorem 2.7, there exist $a_i \in \mathrm{Ker}(f_i)$ and $b_i \in \mathrm{Im}(f_i)$ such that $x = a_i \circ b_i$, $i = 1, 2$. Then

$$\begin{aligned}(f_1 f_2)(x) &= f_1(f_2(x)) = f_1(f_2(a_2 \circ b_2)) \\ &= f_1(f_2(a_2) \circ f_2(b_2)) = f_1(b_2) \\ &= f_1(x \wedge b_2) = f_1((a_1 \wedge b_2) \circ (b_1 \wedge b_2)) \\ &= f_1(a_1 \wedge b_2) \circ f_1(b_1 \wedge b_2) = b_1 \wedge b_2.\end{aligned}$$

Similarly $(f_1 f_2)(x) = b_2 \wedge b_1$. Since $b_1 \wedge b_2 = b_2 \wedge b_1$, $f_1 f_2 = f_2 f_1$, as required. □

PROPOSITION 2.11. *Let X be a S-I-BCK-algebra. Then X is bound -ed if and only if $C(S) = S$.*

PROOF. We denote 1 as the greatest element of X. Suppose that $f \in C(S)$. By $X = \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$, there exist $a \in \mathrm{Ker}(f)$ and $b \in \mathrm{Im}(f)$ such that $1 = a \circ b$. Then for any $x \in X$,

$$\begin{aligned}x * a &= (x \wedge 1) * a = (x \wedge (a \circ b)) * a = ((x \wedge a) \circ (x \wedge b)) * a \\ &= ((x \wedge a) * a) \circ ((x \wedge b) * a) = (x \wedge b) * a \\ &= (x \wedge b) * ((x \wedge b) \wedge a) = (x \wedge b) * 0 = x \wedge b.\end{aligned}$$

Hence

$$\begin{aligned}f(x) &= f(x \wedge 1) = f((x \wedge a) \circ (x \wedge b)) \\ &= f(x \wedge a) \circ f(x \wedge b) = x \wedge b = x * a = f_a(x),\end{aligned}$$

namely, $C(S) \subseteq S$. in addition, $S \subseteq C(S)$ hence $C(S) = S$.

Conversely it is easy to see that the map $\theta : X \to X; x \mapsto 0$ belongs to $C(S)$. Since $C(S) = S$, there exists an element 1 in X such that $\theta = f_1$. Thus 1 is just the greatest element of X since $x * 1 = f_1(x) = \theta(x) = 0$ for all $x \in X$. Hence X is bounded. □

Note that there exist non-bounded S-I-BCK-algebras, we have the following.

COROLLARY 2.12. *Let X be a non-bounded S-I-BCK-algebra. Then $S \subset C(S)$.*

LEMMA 2.13. *Let X be a S-I-BCK-algebra and $f, g \in C(S)$. Then $f = g$ if and only if $Ker(f) = Ker(g)$.*

PROOF. It suffices to prove the part "only if". Let

$$A = \{x \in X \mid x \wedge y = 0 \text{ for all } y \in \mathrm{Ker}(f)\}.$$

Then $\mathrm{Im}(f) \subseteq A$ since $X = \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$. For any $x \in A$, by $x * f(x) \in \mathrm{Ker}(f)$,

$$x * f(x) = x * (x * (x * f(x))) = (x * f(x)) \wedge x = 0,$$

in addition, $f(x) * x = 0$ by Lemma 2.2. Hence $x = f(x) \in \mathrm{Im}(f)$. This means $\mathrm{Im}(f) = A$. With the same as the proof just now, by our hypothesis, we have also $A = \mathrm{Im}(g)$. Hence $\mathrm{Im}(f) = \mathrm{Im}(g)$. Now for any $x = a \circ b \in X$ with $a \in \mathrm{Ker}(f)$ and $b \in \mathrm{Im}(f)$, $f(x) = f(a \circ b) = f(a) \circ f(b) = b = g(b) = g(a) \circ g(b) = g(a \circ b) = g(x)$. This follows $f = g$. □

PROPOSITION 2.14. *Let A be a summand of a S-I-BCK-algebra X. Then there exists f in $C(S)$ such that $Ker(f) = A$.*

PROOF. By A a summand of X, there is an ideal B of X such that $X = A \circ B$. Put

$$f : X \to X;\ x = a \circ b \mapsto b,\ a \in A,\ b \in B.$$

Since the representation $x = a \circ b$, $a \in A$, $b \in B$ is unique, f is a map. For any $x, x' \in X$, there are $x = a \circ b$ and $x' = a' \circ b'$ where $a, a' \in A$ and $b, b' \in B$. Then

$$\begin{aligned}
f(x * x') &= f((a \circ b) * x') = f((a * x') \circ (b * x')) \qquad [\text{ by } (1)\ ] \\
&= f((a * (a \wedge x')) \circ (b * (b \wedge x'))) \qquad [\text{ by } (4)\ ] \\
&= f((a * (a \wedge (a' \circ b'))) \circ (b * (b \wedge (a' \circ b')))) \\
&= f((a * ((a \wedge a') \circ (a \wedge b'))) \circ (b * ((b \wedge a') \circ (b \wedge b')))) \\
&= f((a * (a \wedge a')) \circ (b * (b \wedge b'))) \\
&= f((a * a') \circ (b * b')) \qquad [\text{ by } (4)\ ] \\
&= b * b' = f(a \circ b) * f(a' \circ b') = f(x) * f(x').
\end{aligned}$$

Hence $f \in \mathrm{End}(X)$. Note that $f(x) = f(a \circ b) = b \leq a \circ b = x$, by Lemma 2.2, we have $f \in \mathrm{C}(S)$. Moreover by Proposition 2.4,

$$\begin{aligned}\mathrm{Ker}(f) &= \{(a \circ b) * f(a \circ b) \mid a \in A,\, b \in B\}\\ &= \{(a \circ b) * b \mid a \in A,\, b \in B\}\\ &= \{(a * b) \circ (b * b) \mid a \in A,\, b \in B\}\\ &= \{a \mid a \in A\} = A. \quad \square\end{aligned}$$

PROPOSITION 2.15. *Let X be a S-I-BCK-algebra and*

$$\mathfrak{M} = \{ \textit{all summands of } X\}.$$

Then $C(S)$ and $(\mathfrak{M}, \circ)$ are isomorphic with respect to semigroups, where

$$A \circ B = \{a \circ b \mid a \in A,\, b \in B\}$$

for all A, $B \in \mathfrak{M}$.

PROOF. The fact that the map

$$\phi : \ \mathrm{C}(S) \to \mathfrak{M};\ f \mapsto \mathrm{Ker}(f)$$

is an one-one correspondence is got by Lemma 2.13 and Proposition 2.14. Also for all f, $g \in \mathrm{C}(S)$, since $g(a) \leq a$ and f is isotone (see [6], Proposition 9),

$$(fg)(a \circ b) = f(g(a) \circ g(b)) = f(g(a)) \leq f(a) = 0$$

for all $a \in \mathrm{Ker}(f)$ and $b \in \mathrm{Ker}(g)$, which means $\mathrm{Ker}(f) \circ \mathrm{Ker}(g) \subseteq \mathrm{Ker}(fg)$. Next for any $x \in \mathrm{Ker}(fg)$, by $X = \mathrm{Ker}(g) \oplus \mathrm{Im}(g)$, there is a representation

$$x = a \circ b,\ a \in \mathrm{Ker}(g),\ b \in \mathrm{Im}(g).$$

Then $f(b) = f(g(b)) = (fg)(b) \leq (fg)(a \circ b) = (fg)(x) = 0$, which means

$$\mathrm{Ker}(f) \circ \mathrm{Ker}(g) \supseteq \mathrm{Ker}(fg).$$

Hence $\phi(fg) = \mathrm{Ker}(fg) = \mathrm{Ker}(f) \circ \mathrm{Ker}(g) = \phi(f) \circ \phi(g)$ and so $\mathrm{C}(S)$ is isomorphic to $\mathfrak{M}$. $\square$

COROLLARY 2.16. *Let X be a S-I-BCK-algebra. If A and B are summands of X then so is $A \circ B$.*

COROLLARY 2.17. *Let X be a bounded S-I-BCK-algebra. Then S and $\mathfrak{M}$ are isomorphic with respect to semigroups.*

3. The C(S) extension of S-I-BCK-algebras

PROPOSITION 3.1. *Let X be a S-I-BCK-algebra and $f, g \in C(S)$. Then*

$$X = \Big(Ker(f) \cap Im(g)\Big) \oplus \Big(Ker(f) \cap Ker(g)\Big) \oplus Im(f).$$

PROOF. We need only to prove

$$\mathrm{Ker}(f) = \Big(\mathrm{Ker}(f) \cap \mathrm{Im}(g)\Big) \oplus \Big(\mathrm{Ker}(f) \cap \mathrm{Ker}(g)\Big)$$

but this is an immediate consequence of Lemma 1.8 and Theorem 2.7.

Let X be a S-I-BCK-algebra and $f, g \in \mathrm{C}(S)$. We denote

$$\begin{aligned}\mathrm{Ker}(f * g) &= \mathrm{Ker}(f) \cap \mathrm{Im}(g),\\ \mathrm{Im}(f * g) &= \Big(\mathrm{Ker}(f) \cap \mathrm{Ker}(g)\Big) \oplus \mathrm{Im}(f).\end{aligned}$$

According to Proposition 3.1, we have $X = \mathrm{Ker}(f * g) \oplus \mathrm{Im}(f * g)$ and so by Proposition 2.14 and Proposition 2.15, there exists unique $h \in \mathrm{C}(S)$ such that $\mathrm{Ker}(h) = \mathrm{Ker}(f * g)$. We claim $\mathrm{Im}(h) = \mathrm{Im}(f * g)$, in fact, by Theorem 2.7 and Lemma 1.8,

$$\begin{aligned}\mathrm{Im}(f * g) &= \mathrm{Im}(f * g) \cap X\\ &= \mathrm{Im}(f * g) \cap \Big(\mathrm{Ker}(h) \oplus \mathrm{Im}(h)\Big)\\ &= \mathrm{Im}(f * g) \cap \Big(\mathrm{Ker}(f * g) \oplus \mathrm{Im}(h)\Big)\\ &= \Big((\mathrm{Im}(f * g) \cap \mathrm{Ker}(f * g)\Big) \oplus \Big(\mathrm{Im}(f * g) \cap \mathrm{Im}(h)\Big)\\ &= \mathrm{Im}(f * g) \cap \mathrm{Im}(h) \subseteq \mathrm{Im}(h).\end{aligned}$$

Similarly we have $\mathrm{Im}(h) \subseteq \mathrm{Im}(f * g)$, as claimed. We now define $h = f * g$, then $*$ is just a binary operation on $\mathrm{C}(S)$.

In particular, if the operation $*$ is restricted on S, it is also closed, in fact, we have $f_a * f_b = f_{a*b}$ for all $f_a, f_b \in S$. To see this we need only verify

$$(*) \qquad \mathrm{Ker}(f_a * f_b) = \mathrm{Ker}(f_{a*b})$$

by Lemma 2.13. Now if $x \in \mathrm{Ker}(f_a * f_b) = \mathrm{Ker}(f_a) \cap \mathrm{Im}(f_b)$, then $f_a(x) = 0$ and $f_b(x) = x$ and so

$$\begin{aligned} f_{a*b}(x) &= x * (a * b) = ((x * a) * b) \circ (x \wedge b) \qquad [\text{ by } (5)\ ] \\ &= (f_a(x) * b) \circ (x \wedge b) = x \wedge b = f_b(x) \wedge b \\ &= (x * b) \wedge b = 0. \qquad [\text{ by } (3)\ ] \end{aligned}$$

Hence $x \in \mathrm{Ker}(f_{a*b})$. On the other hand if $x \in \mathrm{Ker}(f_{a*b})$, then $x*(a*b) = 0$. Thus $x \in \mathrm{Ker}(f_a)$ is got by $f_a(x) = x * a \leq x * (a * b) = 0$. Also $x \in \mathrm{Im}(f_b)$ is got by

$$\begin{aligned} x &= x * 0 = x * (x * (a * b)) = (a * b) * ((a * b) * x) \\ &= (a * b) * ((a * x) * b) = (a * (a * x)) * b \\ &= (x \wedge a) * b = f_b(x \wedge a). \end{aligned}$$

Hence $x \in \mathrm{Ker}(f_a * f_b)$. We have proved that $(*)$ holds.

PROPOSITION 3.2. *Let X be a S-I-BCK-algebra and $f_1, f_2 \in C(S)$. Suppose that $a = a_i \circ b_i$ where $a_i \in \mathrm{Ker}(f_i)$ and $b_i \in \mathrm{Im}(f_i)$, $i = 1, 2$. Then*

(a) $f_i|_{A(a)} = f_{a_i}$, $i = 1, 2$;

(b) $(f_1 * f_2)|_{A(a)} = f_1|_{A(a)} * f_2|_{A(a)}$.

Where $f|_{A(a)}$ denotes the restriction of f on the initial section $A(a)$.

PROOF. (a) Suppose that $x = x_i \circ y_i \in A(a)$ where $x_i \in \mathrm{Ker}(f_i)$ and $y_i \in \mathrm{Im}(f_i)$, $i = 1, 2$. Then by $x_i * a_i \in \mathrm{Ker}(f_i)$,

$$\begin{aligned} x_i * a_i &= (x_i * a_i) * 0 = (x_i * a_i) * ((x_i * a_i) \wedge b_i) \\ &= (x_i * a_i) * ((x_i * a_i) * ((x_i * a_i) * b_i)) \\ &= (x_i * a_i) * b_i = x_i * (a_i \circ b_i) = x_i * a = 0 \end{aligned}$$

and so

$$\begin{aligned} f_i|_{A(a)}(x) &= f_i(x) = f_i(x_i \circ y_i) = y_i = y_i * (y_i \wedge a_i) \\ &= y_i * a_i = (x_i \circ y_i) * a_i = x * a_i = f_{a_i}(x), \end{aligned}$$

namely, $f_i|_{A(a)} = f_{a_i}$, $i = 1, 2$.

(b) By Lemma 1.8 and Proposition 3.1, $A(a) = \Big(A(a) \cap \mathrm{Ker}(f_1) \cap \mathrm{Im}(f_2)\Big) \oplus \Big(A(a) \cap \mathrm{Ker}(f_1) \cap \mathrm{Ker}(f_2)\Big) \oplus \Big(A(a) \cap \mathrm{Im}(f_1)\Big)$. Then for any $x \in A(a)$,

$$x = ((x \wedge a_1) * a_2) \circ (x \wedge a_1 \wedge a_2) \circ (x \wedge b_1)$$

where $(x \wedge a_1) * a_2 \in \mathrm{Ker}(f_1 * f_2)$ and $(x \wedge a_1 \wedge a_2) \circ (x \wedge b_1) \in \mathrm{Im}(f_1 * f_2)$. Moreover

$$\begin{aligned} x * (a_1 * a_2) &= x * (a_1 * (a_1 \wedge a_2)) \qquad [\text{ by } (4)] \\ &= ((x * a_1) * (a_1 \wedge a_2)) \circ (x \wedge a_1 \wedge a_2) \qquad [\text{ by } (5)] \\ &= (x * (a_1 \circ (a_1 \wedge a_2))) \circ (x \wedge a_1 \wedge a_2) \\ &= (x * a_1) \circ (x \wedge a_1 \wedge a_2) \\ &= ((x \wedge a) * a_1) \circ (x \wedge a_1 \wedge a_2) \\ &= (((x \wedge a_1) \circ (x \wedge b_1)) * a_1) \circ (x \wedge a_1 \wedge a_2) \\ &= (((x \wedge a_1) * a_1) \circ ((x \wedge b_1) * a_1)) \circ (x \wedge a_1 \wedge a_2) \\ &= (x \wedge b_1) \circ (x \wedge a_1 \wedge a_2). \end{aligned}$$

Then

$$\begin{aligned} (f_1 * f_2)|_{A(a)}(x) &= (f_1 * f_2)(x) = (x \wedge a_1 \wedge a_2) \circ (x \wedge b_1) \\ &= x * (a_1 * a_2) = f_{a_1 * a_2}(x) = (f_{a_1} * f_{a_2})(x) \\ &= (f_1|_{A(a)} * f_2|_{A(a)})(x). \end{aligned}$$

Hence (b) holds. □

THEOREM 3.3. *Let X be a S-I-BCK-algebra. Then $(C(S), *, f_0)$ forms a bounded S-I-BCK-algebra where the operation $*$ is mentioned above and f_0 is the identity map on X.*

PROOF. Suppose that $f, g, f_1, f_2, f_3 \in C(S)$ and $a = a_i \circ b_i \in X$ with $a_i \in \mathrm{Ker}(f_i)$ and $b_i \in \mathrm{Im}(f_i)$, $i = 1, 2, 3$.

(1) $((f_1 * f_2) * (f_1 * f_3)) * (f_3 * f_2) = f_0$ is got by

$$\begin{aligned}
&(((f_1 * f_2) * (f_1 * f_3)) * (f_3 * f_2))(a)\\
=&(((f_1 * f_2) * (f_1 * f_3)) * (f_3 * f_2))|_{A(a)}(a)\\
=&(((f_1|_{A(a)} * f_2|_{A(a)}) * (f_1|_{A(a)} * f_3|_{A(a)})) * (f_3|_{A(a)} * f_2|_{A(a)}))(a)\\
=&(((f_{a_1} * f_{a_2}) * (f_{a_1} * f_{a_3})) * (f_{a_3} * f_{a_2}))(a)\\
=&f_{((a_1 * a_2)*(a_1 * a_3))*(a_3 * a_2)}(a) = f_0(a).
\end{aligned}$$

(2) Since $\mathrm{Im}(f_0) = X$, $\mathrm{Ker}(f * f_0) = \mathrm{Ker}(f) \cap \mathrm{Im}(f_0) = \mathrm{Ker}(f)$, it follows $f * f_0 = f$ by Proposition 2.13.

(3) Because $\mathrm{Ker}(f_0) = \{0\}$, $\mathrm{Ker}(f_0 * f) = \mathrm{Ker}(f_0) \cap \mathrm{Im}(f) = \mathrm{Ker}(f_0)$ and so $f_0 * f = f_0$.

(4) If $f * g = g * f = f_0$, then $\mathrm{Ker}(f) \cap \mathrm{Im}(g) = \mathrm{Ker}(g) \cap \mathrm{Im}(f) = \{0\}$. Hence $\mathrm{Ker}(f) = \mathrm{Ker}(f) \cap X = \Big(\mathrm{Ker}(f) \cap \mathrm{Ker}(g)\Big) \oplus \Big(\mathrm{Ker}(f) \cap \mathrm{Im}(g)\Big) = \mathrm{Ker}(f) \cap \mathrm{Ker}(g)$. Similarly $\mathrm{Ker}(g) = \mathrm{Ker}(g) \cap \mathrm{Ker}(f)$. Hence $\mathrm{Ker}(f) = \mathrm{Ker}(g)$ and $f = g$.

(5) The zero map θ is the greatest element of $C(S)$. Since

$$\mathrm{Ker}(f) \cap \mathrm{Im}(\theta) = \mathrm{Ker}(f) \cap \{0\} = \mathrm{Ker}(f_0),$$

in other hand, $f * \theta = f_0$.

(6) $f_1 * (f_2 * f_1) = f_1$ is got by

$$\begin{aligned}
(f_1 * (f_2 * f_1))(a) &= (f_1 * (f_2 * f_1))|_{A(a)}(a)\\
&= (f_1|_{A(a)} * (f_2|_{A(a)} * f_1|_{A(a)}))(a)\\
&= (f_{a_1} * (f_{a_2} * f_{a_1}))(a) = f_{a_1 * (a_2 * a_1)}(a)\\
&= f_{a_1}(a) = f_1|_{A(a)}(a) = f_1(a).
\end{aligned}$$

(7) First we claim that $(f_1 f_2)|_{A(a)} = f_{a_1 \circ a_2}$, in fact, for any $x \in A(a)$, by $x * a_2 \leq x$, $x * a_2 \in A(a)$. Then

$$\begin{aligned}
(f_1 f_2)|_{A(a)}(x) &= (f_1 f_2)(x) = f_1(f_2(x)) = f_1(f_2|_{A(a)}(x))\\
&= f_1(f_{a_2}(x)) = f_1(x * a_2) = f_1|_{A(a)}(x * a_2)\\
&= (x * a_2) * a_1 = x * (a_1 \circ a_2) = f_{a_1 \circ a_2}(x),
\end{aligned}$$

as claimed. Now $(f_1 f_2) * f_1 \leq f_2$ is got by

$$\begin{aligned}(((f_1 f_2) * f_1) * f_2)(a) &= (((f_1 f_2) * f_1) * f_2)|_{A(a)}(a)\\ &= (((f_1 f_2)|_{A(a)} * f_1|_{A(a)}) * f_2|_{A(a)})(a)\\ &= ((f_{a_1 \circ a_2} * f_{a_1}) * f_{a_2})(a)\\ &= f_{((a_1 \circ a_2) * a_1) * a_2}(a) = f_0(a).\end{aligned}$$

On the other hand if $(f_3 * f_1) * f_2 = f_0$, then

$$\begin{aligned}(f_3 * (f_1 f_2))(a) &= (f_3 * (f_1 f_2))|_{A(a)}(a)\\ &= (f_3|_{A(a)} * (f_1 f_2)|_{A(a)})(a)\\ &= (f_{a_3} * (f_{a_1} f_{a_2}))(a) = f_{a_3 * (a_1 \circ a_2)}(a)\\ &= f_{(a_3 * a_1) * a_2}(a) = ((f_3 * f_1) * f_2)|_{A(a)}(a)\\ &= ((f_3 * f_1) * f_2)(a) = f_0(a)\end{aligned}$$

and hence $f_3 \leq f_1 f_2$.

So far we have already proved that $(C(S), *, f_0)$ is a bounded S-I-BCK-algebra. □

REMARK. We see from Theorem 3.3 that the operation $\circ$ on $C(S)$ is just the composition "$\cdot$" of maps.

PROPOSITION 3.4. *Let X be a S-I-BCK-algebra and $f, g \in C(S)$. Then $f \leq g$ if and only if $Ker(f) \subseteq Ker(g)$.*

PROOF. If $f \leq g$, namely, $f * g = f_0$, then

$$\mathrm{Ker}(f) \cap \mathrm{Im}(g) = \mathrm{Ker}(f * g) = \mathrm{Ker}(f_0) = \{0\}.$$

So $\mathrm{Ker}(f) = \Big(\mathrm{Ker}(f) \cap \mathrm{Ker}(g)\Big) \oplus \Big(\mathrm{Ker}(f) \cap \mathrm{Im}(g)\Big) = \mathrm{Ker}(f) \cap \mathrm{Ker}(g) \subseteq \mathrm{Ker}(g)$.

Conversely by $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(g)$,

$$\mathrm{Ker}(f * g) = \mathrm{Ker}(f) \cap \mathrm{Im}(g) \subseteq \mathrm{Ker}(g) \cap \mathrm{Im}(g) = \{0\} = \mathrm{Ker}(f_0)$$

then $f * g = f_0$ so $f \leq g$. □

THEOREM 3.5. *Let X be a S-I-BCK-algebra. Then*
(a) *S is an ideal of $C(S)$;*
(b) *X is BCK-isomorphic to S;*
(c) *If X is non-bounded then S is a proper ideal of $C(S)$ but not a summand of $C(S)$.*

PROOF. (a) Put $f_a \in S$ and $f \in C(S)$. If $f \leq f_a$, then $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(f_a) = A(a)$ by Proposition 3.4. Note that $A(a)$ has the greatest element a, by

$$A(a) = \Big(A(a) \cap \mathrm{Ker}(f)\Big) \oplus \Big(A(a) \cap \mathrm{Im}(f)\Big),$$

$\mathrm{Ker}(f)$ has also one, say a_1. Then $\mathrm{Ker}(f) = A(a_1) = \mathrm{Ker}(f_{a_1})$. Hence $f = f_{a_1} \in S$. Next clearly $f_a f_b = f_{a \circ b} \in S$ for all $f_a, f_b \in S$, which means that S is an additive ideal of X thus an ideal of X by Proposition 1.3.

(b) It is easy to verify that the map $\psi : X \to S; a \mapsto f_a$ is a BCK-isomorphism from X to S thus X is BCK-isomorphic to S.

(c) If X is non-bounded then so is S by (b) but $C(S)$ is bounded by Theorem 3.3. This means that S must not be a summand of $C(S)$. □

COROLLARY 3.6. *Let X be a S-I-BCK-algebra. If X is non-bounded then X can be imbedded in $C(S)$ such that X is a proper ideal of $C(S)$ but not a summand of $C(S)$.*

THEOREM 3.7. *Let X be a non-bounded S-I-BCK-algebra. Suppose that $N(S) = \{ Nf_a \mid Nf_a = \theta * f_a,\ f_a \in S\}$ and $T = S \cup N(S)$. Then*
(a) *T is a subalgebra of $C(S)$;*
(b) *T itself is also a S-I-BCK-algebra;*
(c) *S is a maximal ideal of T, but not a summand of T.*

PROOF. (a) Assume that $f_a, f_b \in S$. Then

$$\begin{aligned} f_a * f_b &= f_{a*b} \in S \subseteq T, \\ f_a * \mathrm{N} f_b &= f_a * (\theta * f_b) = ((f_a * \theta) * f_b) \cdot (f_a \wedge f_b) \qquad [\text{ by } (5)\ ] \\ &= f_a \wedge f_b = f_b * (f_b * f_a) = f_{b*(b*a)} \in S \subseteq T, \\ \mathrm{N} f_a * f_b &= (\theta * f_a) * f_b = \theta * (f_a f_b) \\ &= \theta * f_{a \circ b} = \mathrm{N} f_{a \circ b} \in \mathrm{N}(S) \subseteq T, \\ \mathrm{N} f_a * \mathrm{N} f_b &= (\theta * f_a) * (\theta * f_b) = (\theta * (\theta * f_b)) * f_a \\ &= (\theta \wedge f_b) * f_a = f_b * f_a = f_{b*a} \in S \subseteq T \end{aligned}$$

and hence T is a subalgebra of $\mathrm{C}(S)$.

(b) If $f_a, f_b \in S$, then

$$\begin{aligned} f_a f_b &= f_{a \circ b} \in S \subseteq T, \\ f_a \cdot \mathrm{N} f_b &= \mathrm{N} f_b \cdot f_a = (\theta * f_b) \cdot f_a = (\theta * f_b) \cdot (\theta * (\theta * f_a)) \\ &= \theta * (f_b \wedge (\theta * f_a)) = \theta * ((f_b \wedge \theta) * f_a)) \qquad [\text{ by } (7)\ ] \\ &= \theta * (f_b * f_a) = \theta * f_{b*a} = \mathrm{N} f_{b*a} \in \mathrm{N}(S) \subseteq T, \\ \mathrm{N} f_a \cdot \mathrm{N} f_b &= (\theta * f_a) \cdot (\theta * f_b) = \theta * (f_a \wedge f_b) \\ &= \theta * (f_b * (f_b * f_a)) = \theta * f_{b*(b*a)} = \mathrm{N} f_{b*(b*a)} \in \mathrm{N}(S) \subseteq T. \end{aligned}$$

Hence T is a S-I-BCK-algebra.

(c) Since S is an ideal of $\mathrm{C}(S)$, it is also an ideal of T. Moreover T has the greatest element θ by $\theta = \theta * f_0 = \mathrm{N} f_0$, but S has not one and so $\theta \notin S$, which means that S is a proper ideal of T. Next for any $\mathrm{N} f_a \in \mathrm{N}(S)$, $f_a \cdot \mathrm{N} f_a \in \langle S, \mathrm{N} f_a \rangle$ which is the ideal generated by the set $S \cup \{\mathrm{N} f_a\}$. Since S is an ideal of T, $\theta \in \langle S, \mathrm{N} f_a \rangle$ is got by $\theta * (f_a \cdot \mathrm{N} f_a) = (\theta * f_a) * \mathrm{N} f_a = \mathrm{N} f_a * \mathrm{N} f_a = f_0$ and hence $\langle S, \mathrm{N} f_a \rangle = T$. This proves that S is a maximal ideal of T. Note that T has the greatest element θ but S has not, we see that S is not a summand of T. □

COROLLARY 3.8. *Let X be a non-bounded S-I-BCK-algebra. Then X can be imbedded in a bounded S-I-BCK-algebra X^* such that X is a maximal ideal of X^* but not a summand of X^*.*

REMARK. In [3] we proved Corollary 3.8 from the way of constructing order dual.

A S-I-BCK-algebra X is called semisimple if every ideal of X is a summand of X (see [2]). We proved that if X is a nonzero semisimple S-I-BCK-algebra, it contains at least a simple ideal ([2], Proposition 3.3).

THEOREM 3.9. *Let X be a nonzero semisimple S-I-BCK-algebra, $\{A_i\}_{i\in I}$ all simple ideal family of X and $A = \prod_{i\in I} A_i = \Big\{ \{x_i\}_{i\in I} \mid x_i \in A_i,\ i \in I\Big\}$. Define a binary operation $*$ on A as $x * y = \{x_i * y_i\}_{i\in I}$ where $x = \{x_i\}_{i\in I}$ and $y = \{y_i\}_{i\in I}$. Then*
(a) *$(A; *, 0)$ is a S-I-BCK-algebra where $0 = \{0_i\}_{i\in I}$;*
(b) *C(S) is BCK-isomorphic to A.*

PROOF. We know that in a S-I-BCK-algebra, any simple ideal A_i contains exactly two elements 0 and e_i and we call its nonzero element e_i as atom. Define

$$\psi : \ \mathrm{C}(S) \to A;\ f \mapsto \{x_i\}_{i\in I},\ x_i = e_i * f(e_i),\ i \in I.$$

(1) ψ is a map since $e_i * f(e_i) \le e_i \in A_i$.
(2) If $\psi(f) = \psi(g)$, then $e_i * f(e_i) = e_i * g(e_i)$ so $f(e_i) = g(e_i)$, $i \in I$. From this we easy to see $f = g$.
(3) For any $x = \{x_i\}_{i\in I} \in A$, let B be the ideal of X generated by the set $\{x_i\}_{i\in I}$ then B is a summand of X by X semisimple. Clearly $e_i \in B$ if and only if $x_i = e_i$ for any atom e_i. Now we choose $f \in \mathrm{C}(S)$ satisfying $\mathrm{Ker}(f) = B$ then if $e_i \in B$, $x_i = e_i = e_i * f(e_i)$, otherwise, $x_i = 0 = e_i * e_i = e_i * f(e_i)$. Hence $\psi(f) = \{x_i\}_{i\in I} = x$.
(4) Suppose that $f, g \in \mathrm{C}(S)$ and e an atom of X. If

$$e \in \mathrm{Ker}(f * g) = \mathrm{Ker}(f) \cap \mathrm{Im}(g)$$

then $f(e) = 0$ and $g(e) = e$ and so $e * (f * g)(e) = e = (e * f(e)) * (e * g(e))$. If $e \in \mathrm{Im}(f * g) = \Big(\mathrm{Ker}(f) \cap \mathrm{Ker}(g)\Big) \oplus \mathrm{Im}(f)$, then when $e \in \mathrm{Ker}(f) \cap \mathrm{Ker}(g)$,

$$e * (f * g)(e) = e * e = 0 = (e * 0) * (e * 0) = (e * f(e)) * (e * f(e))$$

and when $e \in \mathrm{Im}(f)$,

$$e * (f * g)(e) = e * e = 0 = 0 * (e * g(e)) = (e * f(e)) * (e * g(e)).$$

Hence

$$\begin{aligned}\psi(f * g) &= \{e_i * (f * g)(e_i)\}_{i \in I} = \{(e_i * f(e_i)) * (e_i * g(e_i))\}_{i \in I} \\ &= \{e_i * f(e_i)\}_{i \in I} * \{e_i * g(e_i)\}_{i \in I} = \psi(f) * \psi(g).\end{aligned}$$

In sum ψ is a BCK-isomorphism from C(S) to A and hence C(S) is isomorphic to $A = \prod_{i \in I} A_i$. We then obtain at once that $(A, *, 0)$ is a S-I-BCK-algebra. □

COROLLARY 3.10. *Let X be a semisimple S-I-BCK-algebra and A the set of all atoms. Then $\mid C(S) \mid = 2^{|A|}$ where $\mid A \mid$ denotes the cardinal number of A.*

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