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ASYMPTOTIC BEHAVIOR OF IDEALS RELATIVE TO INJECTIVE A-MODULES

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ABSTRACT. This paper is concerned with an asymptotic behavior of ideals relative to injective modules over the commutative Noetherian ring A : under what conditions on A can we show that $\overline{\mathrm{At}^*}(\mathfrak{a}, E) = \mathrm{At}^*(\mathfrak{a}, E)$?

1. Introduction

Let E be an injective module over a commutative Noetherian ring A (with non-zero identity), and let $\mathfrak{a}$ be an ideal of A.

In [1, 2.2], Toroghy and Sharp showed that the submodule $(0 :_E \mathfrak{a})$ of E has a secondary representation, and so we can form the finite set $\mathrm{Att}_A(0 :_E \mathfrak{a})$ of its attached prime ideals. (Accounts of the relevant theory of secondary representation of modules and attached prime ideals are available in [5], [4], and [8], and we shall use the terminology of [12] and [5] for these topics.) One of the main results of [1] is that the sequence of sets

$$(\mathrm{Att}_A(0 :_E \mathfrak{a}^n))_{n\in\mathbb{N}}$$

is ultimately constant: its ultimate constant value denoted by $\mathrm{At}^*(\mathfrak{a}, E)$. This result can be viewed as a companion to [13, (3.1)(iii)], which shows that, for an ideal I in a commutative ring R (with identity) and an Artinian R-module N, the sequence of sets

$$(\mathrm{Att}_R(0 :_N I^n))_{n\in\mathbb{N}}$$

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is ultimately constant; and this result can, in turn, be viewed as dual to Brodmann's result [3] that, for a Noetherian A-module M, the sequence of sets

$$(\mathrm{Ass}_A(M/\mathfrak{a}^n M))_{n\in\mathbb{N}}$$

is ultimately constant.

In [2], Toroghy and Sharp introduced concepts of reduction and integral closure of $\mathfrak{a}$ relative to E, and showed that these concepts have properties which reflect some of those of the classical concepts of reduction and integral closure introduced by Northcott and Rees in [9].

We say that the ideal $\mathfrak{a}$ of A is a *reduction* of the ideal $\mathfrak{b}$ of A *relative to* E if $\mathfrak{a} \subseteq \mathfrak{b}$ and there exists $s \in \mathbb{N}$ such that $(0 :_E \mathfrak{a}\mathfrak{b}^s) = (0 :_E \mathfrak{b}^{s+1})$. An element x of A is said to be *integrally dependent on* $\mathfrak{a}$ *relative to* E if there exists $n \in \mathbb{N}$ such that

$$(0 :_E \sum_{i=1}^{n} x^{n-i}\mathfrak{a}^i) \subseteq (0 :_E x^n).$$

In fact, this is the case if and only if $\mathfrak{a}$ is a reduction of $\mathfrak{a} + Ax$ relative to E [2, 2.2]; moreover,

$$\mathfrak{a}^{*(E)} := \{y \in A \mid y \text{ is integrally dependent on } \mathfrak{a} \text{ relative to } E\}$$

is an ideal of A, called the *integral closure of* $\mathfrak{a}$ *relative to* E, and is the largest ideal of A which has $\mathfrak{a}$ as a reduction relative to E. The main result of [2] is that the sequence of sets

$$(\mathrm{Att}_A(0 :_E (\mathfrak{a}^n)^{*(E)}))_{n\in\mathbb{N}}$$

is increasing and ultimately constant: its ultimate constant value denoted by $\overline{\mathrm{At}}^*(\mathfrak{a}, E)$. The proof of this result used, among other things, the result of Ratliff [10, (2.4) and (2.7)] that the sequence of sets (ass $(\mathfrak{a}^n)^-)_{n\in\mathbb{N}}$ is increasing and ultimately constant, where $(\mathfrak{a}^n)^-$ denotes the classical integral closure of the ideal $\mathfrak{a}^n$.

The above-mentioned results of Brodmann and Ratliff have led to a large body of research: see, for example, McAdam's book [6]. Indeed, that research provides ideas for possible directions in which the theory of asymptotic behavior of ideals relative to injective A-modules might be pursued. For example, [6, 4.7] showed that for an ideal $\mathfrak{a}$ of a 2-dimensional normal Noetherian domain A, $\overline{\mathrm{As}^*}(\mathfrak{a}, A)$ always equals $\mathrm{As}^*(\mathfrak{a}, A)$. This result raise questions about asymptotic behavior relative to E: when $\overline{\mathrm{At}^*}(\mathfrak{a}, E) = \mathrm{At}^*(\mathfrak{a}, E)$? This question is the concern of this paper. The purpose of this paper is to investigate some domains having that property.

2. Notations and previous results

Throughout the remainder of this paper, $\mathfrak{a}$ will denote an ideal of the commutative Noetherian ring A, and E will denote an injective A-module.

NOTATION 2.1. (i) We shall use the notation $\mathrm{Occ}(E)$ of [12, section 2] in connection with our injective A-module E: this is explained as follows. By well-known work of Matlis and Gabriel, there is a family $(P_\alpha)_{\alpha\in\Lambda}$ of prime ideals of A for which $E \cong \oplus_{\alpha\in\Lambda} E(A/P_\alpha)$ (we use $E(L)$ to denote the injective envelope of an A-module L), and the set $\{P_\alpha \mid \alpha \in \Lambda\}$ is uniquely determined by E: we denote it by $\mathrm{Occ}(E)$.

(ii) Let R be a commutative ring with identity, I an ideal of R and let N be a Noetherian R-module, Brodmann [3] (especially) proved that both the sequences of sets

$$(\mathrm{Ass}_R(N/I^nN))_{n\in\mathbb{N}} \quad \text{and} \quad (\mathrm{Ass}_R(I^{n-1}N/I^nN))_{n\in\mathbb{N}}$$

are ultimately constant; let $\mathrm{As}^*(I, N)$ and $\mathrm{Bs}^*(I, N)$ denote their ultimate constant values (respectively).

Let M be an Artinian R-module. In [11], it was proved that both the sequences of sets

$$(\mathrm{Att}_R(0:_M I^n))_{n\in\mathbb{N}} \quad \text{and} \quad (\mathrm{Att}_R((0:_M I^n)/(0:_M I^{n-1})))_{n\in\mathbb{N}}$$

are ultimately constant; let $\mathrm{At}^*(I, M)$ and $\mathrm{Bt}^*(I, M)$ denote their ultimate constant values (respectively).

(iii) We shall use $\mathrm{As}^*(\mathfrak{a}, A)$, $\overline{\mathrm{As}^*}(\mathfrak{a}, A)$, $\mathrm{At}^*(\mathfrak{a}, E)$ and $\overline{\mathrm{At}^*}(\mathfrak{a}, E)$ to denote the ultimate constant values of the sequences of sets

$$(\mathrm{ass}\,\mathfrak{a}^n)_{n\in\mathbb{N}},\ (\mathrm{ass}(\mathfrak{a}^n)^-)_{n\in\mathbb{N}},\ (\mathrm{Att}_A(0:_E \mathfrak{a}^n))_{n\in\mathbb{N}}$$
$$\text{and} \quad (\mathrm{Att}_A(0:_E (\mathfrak{a}^n)^{*(E)}))_{n\in\mathbb{N}}.$$

respectively: references for the results which show that these sequences are all ultimately constant were given in the Introduction.

In the notation of 2.1 (ii), we have $\mathrm{Bs}^*(I,N) \subseteq \mathrm{As}^*(I,N)$ and (by [11, (2.6)], for example) $\mathrm{Bt}^*(I,M) \subseteq \mathrm{At}^*(I,M)$. In the special case in which R itself is Noetherian, McAdam and Eakin [7, Corollary 13] showed that

$$\mathrm{As}^*(I,R)\backslash \mathrm{Bs}^*(I,R) \subseteq \mathrm{Ass}(R),$$

and Sharp [11, (4.2) and (4.3)] adapted their argument to show that

$$\mathrm{As}^*(I,N)\backslash \mathrm{Bs}^*(I,N) \subseteq \mathrm{Ass}(N)$$
$$\text{and } \mathrm{At}^*(I,M)\backslash \mathrm{Bt}^*(I,M) \subseteq \mathrm{Att}(M).$$

In [1], Toroghy and Sharp showed that the obvious analogues of the above-mentioned results for M and I holds for an injective A-module E and an ideal $\mathfrak{a}$ of A.

REMARK 2.2. [6, p.15]. $\overline{\mathrm{As}^*}(\mathfrak{a},A)$ is well behaved with respect to localization. That is, $P \in \overline{\mathrm{As}^*}(\mathfrak{a},A)$ if and only if $PA_s \in \overline{\mathrm{As}^*}(\mathfrak{a}A_s, A_s)$, S multiplicatively closed, $S \cap P = \varnothing$.

We shall need the following results from [1] and [2].

THEOREM 2.3. [1, (2.2)].

$$\mathrm{Att}_A(0:_E \mathfrak{a}) = \{\mathfrak{p} \in \mathrm{ass}\mathfrak{a} \mid \mathfrak{p} \subseteq \mathfrak{q} \textit{ for some } \mathfrak{q} \in \mathrm{Occ}(E)\}.$$

THEOREM 2.4. [1, (3.1)]

$$\mathrm{At}^*(\mathfrak{a},E) = \{\mathfrak{p} \in \mathrm{As}^*(\mathfrak{a},A) \mid \mathfrak{p} \subseteq \mathfrak{q} \textit{ for some } \mathfrak{q} \in \mathrm{Occ}(E)\}.$$

THEOREM 2.5. [1, (3.2)].

$$\mathrm{Bt}^*(\mathfrak{a},E) = \{\mathfrak{p} \in \mathrm{Bs}^*(\mathfrak{a},A) \mid \mathfrak{p} \subseteq \mathfrak{q} \textit{ for some } \mathfrak{q} \in \mathrm{Occ}(E)\}.$$

We use $\mathrm{Bt}^*(\mathfrak{a},E)$ to denote the ultimate constant value of the sequence of sets $(\mathrm{Att}_A((0:_E \mathfrak{a}^{n+1})/(0:_E \mathfrak{a}^n)))_{n\in\mathbb{N}}$.

THEOREM 2.6. [2, (3.2)].

$$\overline{\mathrm{At}^*}(\mathfrak{a}, E) = \{\mathfrak{p} \in \overline{\mathrm{As}^*}(\mathfrak{a}, A) \mid \mathfrak{p} \subseteq \mathfrak{q} \textit{ for some } \mathfrak{q} \in \mathrm{Occ}(E)\}.$$

3. Consequences of results of McAdam

Let E be an injective A-module. Then, by [12, (2.6)],

$$\mathrm{Att}(E) = \{\mathfrak{p} \in \mathrm{Ass}(A) \mid \mathfrak{p} \subseteq \mathfrak{q} \text{ for some } \mathfrak{q} \in \mathrm{Occ}(E)\}.$$

Throughout this section by a minimal prime of E we mean a minimal prime of $\mathrm{Att}(E)$. It is clear that the set of minimal primes of E is a subset of the set of minimal primes of A.

THEOREM 3.1. $\overline{\mathrm{At}^*}(\mathfrak{a}, E) \subseteq \mathrm{At}^*(\mathfrak{a}, E)$. *In fact, if* $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}, E)$, *then either* $\mathfrak{p} \in \mathrm{Bt}^*(\mathfrak{a}, E)$ *or* $\mathfrak{p}$ *is a minimal prime of* E.

PROOF. Let $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}, E)$. Then $\mathfrak{p} \in \overline{\mathrm{As}^*}(\mathfrak{a}, A)$ and $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \mathrm{Occ}(E)$ by (2.6). Now, by [6, 3.17], $\mathfrak{p} \in \mathrm{As}^*(\mathfrak{a}, A)$. Therefore $\mathfrak{p} \in \mathrm{As}^*(\mathfrak{a}, A)$ and $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \mathrm{Occ}(E)$. This implies that $\mathfrak{p} \in \mathrm{At}^*(\mathfrak{a}, E)$ by (2.4) and so $\overline{\mathrm{At}^*}(\mathfrak{a}, E) \subseteq \mathrm{At}^*(\mathfrak{a}, E)$.

On the other hand, if $\mathfrak{p} \in \overline{\mathrm{As}^*}(\mathfrak{a}, A)$, then either $\mathfrak{p} \in \mathrm{Bs}^*(\mathfrak{a}, A)$ or $\mathfrak{p}$ is a minimal prime of A by [6, 3.17]. Hence if $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}, E)$ then, by the above argument, we have either $\mathfrak{p} \in \mathrm{Bs}^*(\mathfrak{a}, A)$ and $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \mathrm{Occ}(E)$ or $\mathfrak{p}$ is a minimal prime of A and $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \mathrm{Occ}(E)$. This implies that either $\mathfrak{p} \in \mathrm{Bt}^*(\mathfrak{a}, E)$ by (2.5) or $\mathfrak{p}$ is a minimal prime of E.

THEOREM 3.2. *Let A be a normal 2-dimensional Noetherian domain. Then*

$$\mathrm{At}^*(\mathfrak{a}, E) = \overline{\mathrm{At}^*}(\mathfrak{a}, E).$$

PROOF. By [6, 4.7], $\mathfrak{p} \in \mathrm{As}^*(\mathfrak{a}, A)$ if and only if $\mathfrak{p} \in \overline{\mathrm{As}^*}(\mathfrak{a}, A)$. Hence it follows that $\mathfrak{p} \in \mathrm{At}^*(\mathfrak{a}, E)$ if and only if $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}, E)$ by (2.4) and (2.6).

THEOREM 3.3. *Assume A is a 2-dimensional local unique factorization domain. Then*

$$\mathrm{At}^*(\mathfrak{a}, E) = \overline{\mathrm{At}^*}(\mathfrak{a}, E) = \mathrm{Att}_E(0 :_E \mathfrak{a})$$

PROOF. Let $\mathrm{As}(\mathfrak{a}, n) = \mathrm{Ass}_A(A/\mathfrak{a}^n)$ and let $\overline{\mathrm{As}}(\mathfrak{a}, n) = \mathrm{Ass}_A(A/(\mathfrak{a}^n)^-)$. Then, by [6, 8.11],

$$\begin{aligned}\mathrm{As}(\mathfrak{a}, 1) = \mathrm{As}(\mathfrak{a}, 2) = \cdots = \mathrm{As}^*(\mathfrak{a}, A) = \overline{\mathrm{As}}(\mathfrak{a}, 1) = \overline{\mathrm{As}}(\mathfrak{a}, 2) \\ = \cdots = \overline{\mathrm{As}^*}(\mathfrak{a}, A)\end{aligned}$$

Now the result follows from (2.3), (2.4) and (2.6).

THEOREM 3.4. *Let A be a locally quasi-unmixed which is also Cohen-Macaulay. Then*

$$\mathrm{At}^*(\mathfrak{a}, E) = \overline{\mathrm{At}^*}(\mathfrak{a}, E)$$

for every ideal $\mathfrak{a}$ of the principal class.

PROOF. By [6, 8.12], $\mathrm{As}^*(\mathfrak{a}, A) = \overline{\mathrm{As}^*}(\mathfrak{a}, A)$ for every ideal $\mathfrak{a}$ of the principal class. Hence the result follows from (2.4) and (2.6).

THEOREM 3.5. *Let $\mathfrak{a}$ and $\mathfrak{p}$ be ideals of A with $\mathfrak{a} \subseteq \mathfrak{p}$. Then the followings are equivalent:*

(i) $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}, E)$;
(ii) $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}\mathfrak{b}, E)$ *for each ideal $\mathfrak{b}$ of A such that, for every minimal prime $\mathfrak{p}'$ of E, $\mathfrak{b} \not\subseteq \mathfrak{p}'$;*
(iii) $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}c, E)$ *for each $c \in A$ not contained in any minimal prime of E;*
(iv) *there exists an element $c \in A$ not contained in any minimal prime of E with $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}c, E)$*

PROOF. (i)$\Longrightarrow$(ii). Let $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}, E)$ and suppose that there exists an ideal $\mathfrak{b}$ of A such that $\mathfrak{b}$ is not contained in any minimal prime of E with $\mathfrak{p} \notin \overline{\mathrm{At}^*}(\mathfrak{a}\mathfrak{b}, E)$. Then, by (2.6), there exists $\mathfrak{q} \in \mathrm{Occ}(E)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Now since $\mathfrak{p} \notin \overline{\mathrm{At}^*}(\mathfrak{a}\mathfrak{b}, E)$, $\mathfrak{p} \notin \overline{\mathrm{As}^*}(\mathfrak{a}\mathfrak{b}, A)$. Therefore, by (2.2), $\mathfrak{p}A_{\mathfrak{q}} \notin \overline{\mathrm{As}^*}(\mathfrak{a}A_{\mathfrak{q}}\mathfrak{b}A_{\mathfrak{q}}, A_{\mathfrak{q}})$. Now if $\mathfrak{b} \not\subseteq \mathfrak{q}$, then $\mathfrak{b}A_{\mathfrak{q}} = A_{\mathfrak{q}}$ and so $\mathfrak{p}A_{\mathfrak{q}} \notin \overline{\mathrm{As}^*}(\mathfrak{a}A_{\mathfrak{q}}, A_{\mathfrak{q}})$. So, by (2.2), $\mathfrak{p} \notin \overline{\mathrm{As}^*}(\mathfrak{a}, A)$. Hence, by (2.6), $\mathfrak{p} \notin \overline{\mathrm{At}^*}(\mathfrak{a}, E)$. But this is a contradiction to hypothesis. Now if $\mathfrak{b} \subseteq \mathfrak{q}$, then $\mathrm{ht}\mathfrak{b}A_{\mathfrak{q}} > 0$. To see this we note that if $\mathrm{ht}\mathfrak{b}A_{\mathfrak{q}} = 0$, then there exists a minimal prime $\mathfrak{p}''A_{\mathfrak{q}}$ of $\mathfrak{b}A_{\mathfrak{q}}$ such that $\mathrm{ht}\mathfrak{p}''A_{\mathfrak{q}} = 0$. So $\mathfrak{p}''A_{\mathfrak{q}}$ is a minimal prime of $A_{\mathfrak{q}}$. This implies that $\mathfrak{p}''$ is a minimal prime of A. But $\mathfrak{p}'' \subseteq \mathfrak{q} \in \mathrm{Occ}(E)$. It follows that $\mathfrak{p}''$ is a minimal prime of E. So by hypothesis, $\mathfrak{b} \not\subseteq \mathfrak{p}''$. But this is a contradiction because $\mathfrak{b}A_{\mathfrak{q}} \subseteq \mathfrak{p}''A_{\mathfrak{q}}$. So $\mathfrak{b}A_{\mathfrak{q}}$ is an ideal of $A_{\mathfrak{q}}$ such that $\mathrm{ht}\mathfrak{b}A_{\mathfrak{q}} > 0$ and $\mathfrak{p}A_{\mathfrak{q}} \notin \overline{\mathrm{As}^*}(\mathfrak{a}A_{\mathfrak{q}}\mathfrak{b}A_{\mathfrak{q}}, A_{\mathfrak{q}})$. Hence, by [6, 3.26], $\mathfrak{p}A_{\mathfrak{q}} \notin \overline{\mathrm{As}^*}(\mathfrak{a}A_{\mathfrak{q}}, A_{\mathfrak{q}})$. This implies that $\mathfrak{p} \notin \overline{\mathrm{As}^*}(\mathfrak{a}, A)$ by (2.2). It follows that $\mathfrak{p} \notin \overline{\mathrm{At}^*}(\mathfrak{a}, E)$. Again this is a contradiction by hypothesis.

(ii)$\Longrightarrow$(iii). Suppose that $c \in A$ and c is not contained in any minimal prime of E. Then for every minimal prime $\mathfrak{p}'$ of E, $cA \not\subseteq \mathfrak{p}'$. So by (ii), $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}cA, E) = \overline{\mathrm{At}^*}(\mathfrak{a}c, E)$.

(iii)$\Longrightarrow$(iv).This is clear.

(iv)$\Longrightarrow$(i). Suppose that there exists an element c not contained in any minimal prime of E with $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}c, E)$. Then, by (2.6), $\mathfrak{p} \in \overline{\mathrm{As}^*}(\mathfrak{a}c, A)$ and $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \mathrm{Occ}(E)$. So by (2.2),

$$\mathfrak{p}A_{\mathfrak{q}} \in \overline{\mathrm{As}^*}((\mathfrak{a}c)A_{\mathfrak{q}}, A_{\mathfrak{q}}) = \overline{\mathrm{As}^*}(\mathfrak{a}A_{\mathfrak{q}}\frac{c}{1}A_{\mathfrak{q}}, A_{\mathfrak{q}}).$$

Now we claim that $\frac{c}{1} \in A_{\mathfrak{q}}$ is not contained in any minimal prime of $A_{\mathfrak{q}}$. To see this, suppose $\mathfrak{p}''A_{\mathfrak{q}}$ (where $\mathfrak{p}'' \in \mathrm{Spec}(A)$ and $\mathfrak{p}'' \subseteq \mathfrak{q}$) is a minimal prime of $A_{\mathfrak{q}}$. Then we have $\mathfrak{p}'' \in \mathrm{Ass}(A)$ and $\mathfrak{p}'' \subseteq \mathfrak{q} \in \mathrm{Occ}(E)$. Therefore, by [12, (2.6)], $\mathfrak{p}'' \in \mathrm{Att}(E)$. Moreover, $\mathfrak{p}''$ is a minimal prime of E because $\mathfrak{p}''$ is a minimal prime of A. Therefore, by hypothesis, $c \notin \mathfrak{p}''$ and so $\frac{c}{1} \notin \mathfrak{p}''A_{\mathfrak{q}}$. This implies that $\mathfrak{p}A_{\mathfrak{q}} \in \overline{\mathrm{As}^*}(\mathfrak{a}A_{\mathfrak{q}}, A_{\mathfrak{q}})$ by [6, 3.26]. So by (2.2), $\mathfrak{p} \in \overline{\mathrm{As}^*}(\mathfrak{a}, A)$. Now $\mathfrak{p} \in \overline{\mathrm{As}^*}(\mathfrak{a}, A)$ and $\mathfrak{p} \subseteq \mathfrak{q} \in \mathrm{Occ}(E)$. This implies that $\mathfrak{p} \in \overline{\mathrm{At}^*}(\mathfrak{a}, E)$ by (2.6) and the proof is complete.

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