J. Korean Math. Soc. **32** (1995), No. 4, pp. 835–848

TOTALLY REAL SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR IN A COMPLEX SPACE FORM*

U-HANG KI, BYUNG HAK KIM AND HE-JIN KIM

1. Introduction

Let $M_n(c)$ be an n-dimensional complete and simply connected Kählerian manifold of constant holomorphic sectional curvature c, whi ch is called a *complex space form.* Then according to $c > 0, c = 0$ or $c < 0$ it is a complex projective space P_nC, a complex Euclidean space C^n or a complex hyperbolic space H_nC. An n-dimensional submanifold M in a complex space form $M_n(c)$ with complex structure J is said to be *totally real* if it satisfies $J_x(T_xM) = N_xM$ at any point x in M, where T_xM(resp. N_xM) denotes the tangent space (resp. the normal space) of M at x in M. Totally real submanifolds in a complex space form are studied by many authors from various points of view (for examples :[1],[6],[10] and so on). In particular, for a compact minimal totally real submanifold M in P_nC a sufficient condition for M to become totally geodesic is first given by Chen and Ogiue [1]. Let S be the square of the length of the second fundamental form α of M. Then they proved the following

THEOREM A [1]. *Let M be an n-dimensional compact totally real submanifold in P_nC. If M is minimal and if it satisfies*

$$S < \frac{n(n+1)}{4(2n-1)}c, \tag{1.1}$$

then M is totally geodesic.

Now, it was pointed by Ludden, Okumura and Yano [6] that the estimate (1.1) of the square norm S is best possible. The counter example of the upper bound was also characterized in [6] as follows.

Received October 31, 1994.
1991 AMS Subject Classification: 53B25.
Key words: Totally real submanifolds, complex space form, totally geodesic.
* Supported by TGRC-KOSEF.

THEOREM B. *Let M be an n-dimensional compact totally real submanifold in P_nC. If M is minimal and if it satisfies*

$$S = \frac{n(n+1)}{4(2n-1)}c, \tag{1.2}$$

then $n = 2$ and M is $S^1 \times S^1$ in P_2C.

We denote by $\mathbf{h}$ the mean curvature vector on M and H be the mean curvature of M i.e., the norm of the mean curvature vector $\mathbf{h}$.

The purpose of this paper is to investigate the similar problem to the above theorems for complete totally real submanifolds with parallel mean curvature vector in a complex space form and we obtain the followings.

THEOREM 1. *Let M be an n-dimensional complete totally real submanifold with parallel mean curvature vector $\mathbf{h}$ in a complex projective space P_nC of constant holomorphic sectional curvature c. If there exist some H_0 such that $H < H_0$ and $S_1 \le S \le supS < S_2$, then M is minimal. Consequently M is totally geodesic.*

THEOREM 2. *Let M be an n-dimensional complete totally real submanifold with parallel mean curvature vector $\mathbf{h}$ in a complex hyperbolic space H_nC of constant holomorphic sectional curvature c. Assume that $H^2 < -\frac{1}{4}c$. Then there exists a constant S_1 so that if $\sup S < S_1$, then M is totally geodesic.*

2. Preliminaries

Throughout this paper all manifolds are assumed to be smooth, connected without boundary. We discuss in smooth category. In this section we recall fundamental properties for totally real submanifolds in a complex space form. A complete and simply connected Kählerian manifold of constant holomorphic sectional curvature is called *a complex space form.* We denote by $M_n(c)$ an n-dimensional complex space form of constant holomorphic sectional curvature c. It consists of a complex projective space P_nC, a complex Euclidean space C^n and a complex hyperbolic space H_nC. Let J be the complex structure of $M_n(c)$. We choose the local field of orthonormal frame $e_1, \dots, e_n, e_{n+1} = Je_1, \dots,$

$e_{2n} = Je_n$ adapted to the Kählerian metric of $M_n(c)$ and the dual coframe $\omega_1, \ldots, \omega_{2n}$. The components J_{AB} of the complex structure $J = \sum J_{AB}\omega_A \otimes \omega_B$ with respect to the frame field are given by

$$J_{AB} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{2.1}$$

Then connection forms $\{\omega_{AB}\}$ of $M_n(c)$ are characterized by the structure equations

$$\begin{cases} d\omega_A + \sum \omega_{AB} \wedge \omega_B = 0, \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} + \sum \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \\ \Omega_{AB} = -\frac{1}{2} \sum R'_{ABCD}\omega_C \wedge \omega_D, \end{cases} \tag{2.2}$$

(2.3)
$$R'_{ABCD} = \frac{c}{4}(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD} + J_{AD}J_{BC} - J_{AC}J_{BD} - 2J_{AB}J_{CD}),$$

where Ω_{AB}(resp. R'_{ABCD}) denotes the Riemannian curvature form(resp. the components of the Riemannian curvature tensor R') of $M_n(c)$. In the sequel, the following convention on the range of indices is used, unless otherwise stated:

$$1 \leq A, B, \cdots \leq 2n; \quad 1 \leq i, j, \cdots \leq n; \quad n+1 \leq \alpha, \beta, \cdots \leq 2n.$$

We agree that the repeated indices under a summation sign without indication are summed over the respective range.

Let M be a real n-dimensional submanifold of an n-dimensional complex space form $M_n(c)$. The submanifold M is said to be *totally real* if it satiesfies $J_x(T_xM) = N_xM$ at any point x in M, where T_xM(resp. N_xM) denotes the tangent space (resp. the normal space) of M at x in M.

In this paper we assume that M is an n-dimensional totally real submanifold in $M_n(c)$. Then we can choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1} = Je_1, \ldots, e_{2n} = Je_n$ adapted to the Kählerian metric of $M_n(c)$ and the dual coframe $\omega_1, \ldots, \omega_{2n}$ in such a way that, restricted to the submanifold $M, e_1, \ldots, e_n$ are tangent to M. The canonnical forms $\{\omega_A\}$ and the connection forms $\{\omega_{AB}\}$ restricted to M are also denoted by the same symbols. We then have

$$\omega_\alpha = 0 \qquad for \qquad \alpha = n+1, \dots, 2n. \tag{2.4}$$

We see that $e_1, \dots, e_n$ is a local field of orthonormal frames adapted to the induced Riemannian metric on M and $\omega_1, \dots, \omega_n$ is a local field of its dual coframe on M . It follows from (2.2), (2.4) and Cartan's lemma that we have

$$\omega_{\alpha i} = \sum h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.5}$$

In particular, since M is totally real, we get

$$h^{n+k}_{ij} = h^{n+i}_{jk} = h^{n+j}_{ki} \tag{2.6}$$

The connection froms $\{\omega_{ij}\}$ of M are characterized the structure equations

$$\begin{cases} d\omega_i + \sum \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ \Omega_{ij} = -\frac{1}{2} \sum R_{ijkl}\omega_k \wedge \omega_l, \end{cases} \tag{2.7}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor R) of M. Therefore, from (2.1), (2.2), (2.3) and (2.7), the Gauss equation is given by

$$R_{ijkl} = \frac{c}{4}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + \sum(h^\alpha_{il}h^\alpha_{jk} - h^\alpha_{ik}h^\alpha_{jl}). \tag{2.8}$$

The components of the Ricci curvature Ric and the scalar curvature r are given by

$$R_{jk} = \frac{n-1}{4}c\delta_{jk} + \sum h^\alpha_{ii}h^\alpha_{jk} - \sum h^\alpha_{ji}h^\alpha_{ik}, \tag{2.9}$$

$$r = \frac{n(n-1)}{4}c + \sum\{h^\alpha_{ii}h^\alpha_{jj} - (h^\alpha_{ij})^2\}. \tag{2.10}$$

We also have the structure equation for the normal bundle :

$$(2.11)\qquad \begin{cases} d\omega_\alpha + \sum \omega_{\alpha\beta} \wedge \omega_\beta = 0, \\ d\omega_{\alpha\beta} + \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} = -\frac{1}{2} \sum R_{\alpha\beta ij} \omega_i \wedge \omega_j, \end{cases}$$

where we have again by (2.1),(2.2), (2.3) and (2.7) the Ricci equation

$$(2.12)\qquad R_{\alpha\beta ij} = \frac{1}{4} c(J_{\alpha j} J_{\beta i} - J_{\alpha i} J_{\beta j}) - \sum (h^\alpha_{il} h^\beta_{jl} - h^\alpha_{jl} h^\beta_{il}).$$

The second fundamental form α and the mean curvature vector $\mathbf{h}$ of M are defined by

$$(2.13)\qquad \alpha = \sum h^\alpha_{ij} \omega_i \omega_j e_\alpha, \quad \mathbf{h} = \frac{1}{n} \sum_\alpha \Big(\sum_i h^\alpha_{ii} \Big) e_\alpha.$$

The mean curvature H is defined by

$$(2.14)\qquad H = |\mathbf{h}| = \frac{1}{n} \sqrt{\sum_\alpha \Big(\sum_i h^\alpha_{ii} \Big)^2}.$$

Let $S = \sum (h^\alpha_{ij})^2$ denote the square of the length of the second fundamental form α of M. The components h^α_{ijk} and h^α_{ijkl} of the covariant differentials $\nabla\alpha$ and $\nabla^2\alpha$ of the second fundamental form are defined by

$$(2.15)\qquad \sum h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} - \sum h^\alpha_{kj} \omega_{ki} - \sum h^\alpha_{ik} \omega_{kj} - \sum h^\beta_{ij} \omega_{\beta\alpha},$$

$$(2.16)\qquad \begin{aligned} &\sum h^\alpha_{ijkl} \omega_l \\ =&dh^\alpha_{ijk} - \sum h^\alpha_{ljk} \omega_{li} - \sum h^\alpha_{ilk} \omega_{lj} - \sum h^\alpha_{ijl} \omega_{lk} - \sum h^\beta_{ijk} \omega_{\beta\alpha}, \end{aligned}$$

respectively. The Codazzi equation and the Ricci formula for the second fundamental form are given by

$$(2.17)\qquad h^\alpha_{ijk} - h^\alpha_{ikj} = 0,$$

$$(2.18)\qquad h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum h^\alpha_{im} R_{mjkl} - \sum h^\alpha_{mj} R_{mikl} - \sum h^\beta_{ij} R_{\beta\alpha kl}.$$

The Laplacian $\triangle h_{ij}^{\alpha}$ of the components h_{ij}^{α} of the second fundamental form α is given by $\triangle h_{ij}^{\alpha} = \sum_k h_{ijkk}^{\alpha}$. From (2.17) and (2.18) we get

(2.19)

$$\triangle h_{ij}^{\alpha} = \sum_k h_{kkij}^{\alpha} - \sum h_{km}^{\alpha} R_{mijk} - \sum h_{mi}^{\alpha} R_{mkjk} - \sum h_{ki}^{\beta} R_{\beta\alpha jk}.$$

We put $S_{\alpha\beta} = \sum h_{ij}^{\alpha} h_{ij}^{\beta}$ for any indices α and β and we denote by $(S_{\alpha\beta})$ an $n \times n$ symmetric matrix. It can be assumed to be diagonalizable for a suitable choice of $e_{n+1}, \cdots, e_{2n}$. Set $S_{\alpha} = S_{\alpha\alpha}$. We then have $S = \sum S_{\alpha}$. Next, for any matrix $A = (a_{ij})$, we define $N(A) = tr(A^t A)$. Combining with the Gauss equation (2.8),(2.12) and (2.19) it follows that we get

(2.20)

$$\begin{aligned}\frac{1}{2}\triangle S =& |\nabla\alpha|^2 + \sum h_{ij}^{\alpha} h_{kkij}^{\alpha} + \frac{n+1}{4} cS - \frac{n^2}{2} cH^2 - \sum (S_{\alpha})^2 \\ &+ \sum trH^{\alpha} trH^{\alpha}(H^{\beta})^2 - \sum N(H^{\alpha}H^{\beta} - H^{\beta}H^{\alpha}),\end{aligned}$$

where H^{α} denotes an $n \times n$ symmetric matrix (h_{ij}^{α}) for any index α.

The following generalized maximum principle due to Omori [8] and Yau [12] will play an important role in this paper.

THEOREM 2.1. *Let M be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2-function bounded from above on M, then for any $\varepsilon > 0$, there exists a point p in M such that*

$$F(p) + \varepsilon > supF, \quad |gradF|(p) < \varepsilon, \qquad \triangle F(p) < \varepsilon.$$

3. Minimal totally real submanifolds

This section is concerned with minimal totally real submanifolds in P_nC. Let M be an n-dimensional complete totally real submanifold in a complex projective space P_nC of constant holomorphic sectional curvature c. We assume tnat M is minimal.Then by (2.20) we have

$$(3.1)\quad \frac{1}{c}\triangle S = |\nabla\alpha|^2 + \frac{n+1}{4} cS - \sum (S_{\alpha})^2 - \sum_{\alpha\neq\beta} N(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha}).$$

In order to estimate the last term bounded from below, we need the following lemma due to Chern, do Carmo and Kobayashi [3].

LEMMA 3.1. *Let A and B be $n \times n$ symmetric matrices. Then we have*

$$N(AB - BA) \leq 2N(A)N(B), \tag{3.2}$$

where the equality holds for non-zero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of the following matrices.

$$\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & \\ \hline & & 0 \end{array}\right)$$

Moreover, if A_1, A_2 and A_3 are $n \times n$ symmetric matrices and if

$$N(A_iA_j - A_jA_i) = 2N(A_i)N(A_j)$$

for any distinct indices i and j $(i, j = 1, 2, 3)$, then at least one of the matrices A_i must be zero.

By (3.1) and Lemma 3.1 we have

$$\frac{1}{2}\Delta S \geq \frac{n+1}{4}cS - \sum (S_\alpha)^2 - 2\sum_{\alpha \neq \beta} S_\alpha S_\beta. \tag{3.3}$$

Using this inequality we can prove the following

THEOREM 3.2. *Let M be an n-dimensional complete totally real submanifold in P_nC. If M is minimal and if it satisfies $\sup S < \frac{n(n+1)}{4(2n-1)}c$, then M is totally geodesic.*

Proof. First the function σ_1 and σ_2 are defined by

$$n\sigma_1 = \sum S_\sigma = S, \quad n(n-1)\sigma_2 = 2\sum_{\alpha<\beta} S_\alpha S_\beta$$

Then we get

$$\sum (S_\alpha)^2 = n(\sigma_1)^2 + n(n-1)\{(\sigma_1)^2 - \sigma_2\}, \tag{3.4}$$

$$\sum_{\alpha<\beta} (S_\alpha - S_\beta)^2 = n^2(n-1)\{(\sigma_1)^2 - \sigma_2\}. \tag{3.5}$$

Hence, using (3.4) and (3.5) we obtain

$$\sum (S_\alpha)^2 + 2\sum_{\alpha\neq\beta} S_\alpha S_\beta = -\sum (S_\alpha)^2 + 2\Big(\sum S_\alpha\Big)^2 \leq \Big(2 - \frac{1}{n}\Big)S^2, \tag{3.6}$$

where the equality holds if and only if $(\sigma_1)^2 = \sigma_2$, that is $S_\alpha = S_\beta$ for any distinct indices α and β. Thus we have

$$\frac{1}{2}\Delta S \geq -\frac{2n-1}{n}S^2 + \frac{n+1}{4}cS, \tag{3.7}$$

where the equality holds on M if and only if the second fundamental form is parallel and the equality in (3.2) and (3.6) holds on M.

On the other hand, for any symmetric matrix $A = (a_{ij})$ of order $n(\geq 2)$ it is seen by Hineva [5] that we have

$$a_{ii}a_{jj} - (a_{ij})^2 \geq -\frac{1}{2}\mathrm{tr}A^2.$$

for any distinct indices i and j. Accordingly, for a fixed index α we get

$$h^\alpha_{ii}h^\alpha_{jj} - (h^\alpha_{ij})^2 \geq -\frac{1}{2}\mathrm{tr}(H^\alpha)^2 \geq -\frac{1}{2}S,$$

which yields by (2.9)

$$R_{ij} \geq \Big\{\frac{n-1}{4}c - \frac{n}{2}S\Big\}\delta_{ij},$$

and hence the Ricci curvature on M is bounded from below. Since the square norm S is also bounded by the assumption of the theorem, we can apply the generalized maximum principle (Theorem 2.1) to S. For any given positive number ϵ, there exists a point p at which S satisfies

$$\sup S < S(p) + \epsilon, \quad |\mathrm{grad} S|(p) < \epsilon, \quad \Delta S(p) < \epsilon. \tag{3.8}$$

Consequently (3.7) is reduced to the following relationship

$$\frac{1}{2}\epsilon > -\frac{2n-1}{n}S^2(p) + \frac{n+1}{4}cS(p). \tag{3.9}$$

For a convergent sequence $\{\epsilon_m\}$ such that $\epsilon_m \to 0 (m \to \infty)$ and $\epsilon_m > 0$, there exists a point sequence $\{p_m\}$ such that $\{S(p_m)\}$ converges to $S_0 = \sup S$ by (3.8). On the other hand, it follows from (3.9) that we have

$$\frac{1}{2}\epsilon_m > -\frac{2n-1}{n}S^2(p_m) + \frac{n+1}{4}cS(p_m).$$

Thus we get

$$S_0\Big\{-\frac{2n-1}{n}S_0 + \frac{n+1}{4}c\Big\} \le 0,$$

which means $S_0 = \sup S = 0$ by the assumption. Thus M is totally geodesic.

REMARK. Theorem 3.2 is the complete and non-compact version of Theorem A.

THEOREM 3.3. *Let M be an $n(\ge 2)$-dimensional complete totally real submanifold in P_nC. If M is minimal and if it satisfies $S = \frac{n(n+1)}{4(2n-1)}c$, then $n = 2$ and M is $S^1 \times S^1$ in P_nC.*

Proof. Since S is constant, we have $\Delta S = 0$ and hence by (3.7) we have

$$N(H^\alpha H^\beta - H^\beta H^\alpha) = 2N(H^\alpha)N(H^\beta), \tag{3.10}$$

$$n(n-1)\{(\sigma_1)^2 - \sigma_2\} = 0, S_\alpha = S_\beta \tag{3.11}$$

for any distinct indices α and β. By Lemma 3.1 the equation (3.10) implies that at most two of the H^α's are non-zero. However (3.11) means that if there is a zero matrix H^α, then all matrices H^β are zero and hence $S = 0$. This means that if $n \geq 3$, that M is totally geodesic, a contradiction. Accordingly, the conclusion $n = 2$ is given.

Next we suppose there exist non-zero matrices H^α's. Then, again by Lemma 3.1 there are exactly two H^α's different from the zero matrix and we may suppose that

$$H^3 = \lambda \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right),$$

$$H^4 = \mu \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right),$$

where λ and μ are constant, since the fundamental form is parallel. Thus, by a theorem due to Ludden, Okumura and Yano [6], the proof is complete.

4. Totally real submanifolds with parallel mean curvature vector

This section is devoted to the investigation about totally real submanifolds with parallel mean curvature vector. Let M be an $n(\geq 2)$-dimensional totally real submanifold with parallel mean curvature vector $\mathbf{h}$ in a complex space form $M_n(c)$. Because the mean curvature vector is parallel, the mean curvature H is constant. Suppose $H \neq 0$. Then we can choose e_{n+1} in such a way that its direction coincides with that of the mean curvature vector. Then it is easily seen that we have

$$\text{(4.1)} \qquad w_{\alpha n+1} = 0, H = \text{constant},$$

$$\text{(4.2)} \qquad \text{tr}H^{n+1} = nH, \text{tr}H^\alpha = 0$$

for any index $\alpha(\neq n+1)$. We here notice that for a submanifold M with parallel mean curvature vector $\mathbf{h}$ in $M_n(c)$, it satisfies $S \geq nH^2$, where the equality holds on M if and only if M is totally umbilic.

Proof of Theorem 1. First M is supposed to be minimal. Then by Theorem 3.2, the theorem is already verified. Accordingly we may suppose $H \neq 0$ and we can choose the frame field treated above. Taking account of (2.20), (3.6) and (4.3), we have

$$(4.3)\quad \frac{1}{2}\Delta S \geq -\Big(2-\frac{1}{n}\Big)S^2+\frac{n+1}{4}cS-\frac{n^2}{2}cH^2+nH\sum \mathrm{tr}H^{n+1}(H^\alpha)^2.$$

we estimate the last term bounded from below.

Since H^{n+1} is symmetric matrix, we can choose $\{e_1, \cdots, e_n\}$ such that

$$h_{ij} = \lambda_i \delta_{ij}.$$

In order to estimate the last term of the above equation, we define

$$P_{xyz} = \sum h_{ij}^x J_y{}^i J_z{}^i.$$

Then we can easily find $P_{n+1n+1n+1} \leq (\sum \lambda_i^2)^{1/2}$ by using Cauchy-Schwarz's inequality and the property of the complex structure J.

On the other hand, it is well known [7] that

$$\mathrm{tr}(H^{n+1})^2 = nHP_{n+1n+1n+1} + \frac{c}{4}(n-1).$$

Using above properties, we obtain

$$\frac{nH-\sqrt{D}}{2} \leq \Big(\sum \lambda_i^2\Big)^{1/2} \leq \frac{nH+\sqrt{D}}{2},$$

where $D = n^2H^2 + c(n-1) \geq 0$.

Therefore we have

$$\sum \mathrm{tr}H^{n+1}(H^\alpha)^2 \geq -\frac{nH+\sqrt{D}}{2}\mathrm{tr}(H^\alpha)^2 = -\frac{nH+\sqrt{D}}{2}S.$$

Thus we have

(4.4)

$$\frac{1}{2}\Delta S \geq -\Big[\Big(2-\frac{1}{n}\Big)S^2 + \Big\{\frac{n^2H^2}{2}+\frac{nH\sqrt{n^2H^2+(n-1)c}}{2}-\frac{n+1}{4}c\Big\}S+\frac{n^2}{2}cH^2\Big]$$

Now we consider the equation
(4.5)

$$f(x) = \frac{2n-1}{n}x^2 + \left\{\frac{n^2H^2}{2} + \frac{nH\sqrt{n^2H^2+(n-1)c}}{2} - \frac{n+1}{4}c\right\}x + \frac{n^2}{2}cH^2$$

Then we have
(4.6)

$$f(nH^2) = nH^2\left[(2n-1)H^2 + \frac{n^2H^2}{2} + \frac{nH\sqrt{n^2H^2+(n-1)c}}{2} + \frac{n-1}{4}c\right],$$

(4.7)

$$f(n^2H^2) = n^2H^2\left[\frac{n(5n-2)}{2}H^2 + \frac{nH\sqrt{n^2H^2+(n-1)c}}{2} - \frac{n-1}{4}c\right].$$

Put $f(n^2H^2) = n^2H^2g(H)$, where $g(H) = \left[\frac{n(5n-2)}{2}H^2 + \frac{nH\sqrt{n^2H^2+(n-1)c}}{2} - \frac{n-1}{4}c\right]$. Since $g(0) = -\frac{n-1}{4}c < 0$ and $g(H)$ is monotone increasing function, there exists some constant H_0 such that $g(H_0) = 0$. By the assumption of the theorem and above statements there exist two roots S_1 and S_2 so that $nH^2 < S_1 < S_2$ and the function f is negative on (S_1, S_2). Making use of the same argument as that in the proof of Theorem 3.2 and 3.3 we have the conclusion that $n \geq 3$ and M is totally geodesic.

Proof of Theorem 2. In order to estimate the last term of equation (4.3) in the case of $c < 0$, we need the following property which is proved by cheng and Choi [2].

LEMMA. *Let $a_1, \cdots, a_n$ be bounded real numbers satisfying $\sum a_i = 0$ $\sum(a_i)^2 = a^2 (a > 0)$ and $b_1, \cdots, b_n$ be also real numbers satisfying $\sum(b_j)^2 = b^2 (b > 0)$. Then we have*

$$|\sum a_i(b_i)^2| \leq \sqrt{\frac{n-1}{n}}ab^2,$$

where the equality holds if and only if the $n-1$ a_i are equal to $\mp\sqrt{\frac{1}{n(n-1)}}a$ with each other and the corresponding $n-1$ b_i are equal to 0.

According to the above lemma we have

$$\begin{aligned}\mathrm{tr}H^{n+1}(H^{\alpha})^2 &= \sum_i \lambda_i \sum_k h_{ik}^{\alpha} h_{ik}^{\alpha}\\ &= \sum_{i,j}(\lambda_{ij} - H)(h_{ij}^{\alpha})^2 + \sum_{i,j} H(h_{ij}^{\alpha})^2\\ &\geq \left(H - \sqrt{\frac{n^2-1}{n}}\sqrt{S - nH^2}\right)\sum_{i,j}(h_{ij}^{\alpha})^2,\end{aligned}$$

where we have put $\lambda_i = \lambda_{ij}$.

Thus we have

$$\begin{aligned}\frac{1}{2}\Delta S \geq -\Big\{\Big(2-\frac{1}{n}\Big)S^2 + nH\sqrt{\frac{n^2-1}{n}}\sqrt{S-nH^2}S\\ -\Big(nH^2 + \frac{n+1}{4}c\Big)S + \frac{n^2}{2}cH^2\Big\}.\end{aligned} \tag{4.8}$$

As the same method in proof of Theorem 1, we consider the equation

$$\begin{aligned}f(x) =&\Big(2-\frac{1}{n}\Big)x^2 + \sqrt{n(n^2-1)}H\sqrt{x-nH^2}x\\ &-\Big(nH^2 + \frac{n+1}{4}c\Big)x + \frac{n^2}{2}cH^2.\end{aligned} \tag{4.9}$$

Then we have

$$f(nH^2) = n(n-1)H^2\Big(H^2 + \frac{1}{4}c\Big).$$

Therefore by the assumption there is the least root S_1 greater than nH^2 of (4.9) and the function f is negative on the interval (nH^2, S_1). So we get $\sup S = nH^2$, because of $S \geq nH^2$, which implies $S = nH^2$. This means that M is totally umbilic so we concludes M is totally geodesic using the theorem of Yano and Kon [11].

References

1. B. Y. Chen and K. Ogiue, *On totally real submanifolds*, Trans. Amer. Math. Soc. **193** (1974), 251-266.
2. Q. M. Cheng and S. M. Choi, *Complete space-like submanifolds with parallel mean curvature vector of an indefinite space form*, Tsukuba J. Math. **17** (1993).
3. S. S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, in "Functional analysis and related fields Proc. Conf. in honor of Marshall stone, " Springer-Verlag, Berlin, Heidelberg, New York (1970), 59-75.
4. H. Gauchman, *Pinching problems for totally real minimal submanifolds in $CP^m(c)$*, Tohoku Math. J. **14** (1989), 249-257.
5. S. Hineva, *Submanifolds and the second fundamental tensor*, "Global differential geometry and global analysis, Lecture Notes in Mathematics, No. 1154," Springer-Verlag, Berlin, Heidelberg, New York (1984), 194-203.
6. G. D. Ludden, M. Okumura and K. Yano, *A totally real surfaces in CP^2 that is not totally geodesic*, Proc. Amer. Math. Soc. **53** (1975), 186-200.
7. U. H. Ki and Y. H. Kim, *Totally real submanifolds of a complex space form*, (to appear) in Int. J. Math. and Mathematical sci..
8. H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205-214.
9. H.-F. Sun, *Totally real pseudo-umbilic submanifolds of a complex projective space*, preprint.
10. C. Xia, *On the minimal submanifolds in $CP^m(c)$ and $S^N(1)$*, Kodai Math. J. **15** (1992), 141-153.
11. K. Yano and M. Kon, *"Structures on manifolds," World scientific*, Singapore. (1984).
12. S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure and Appl. Math. **28** (1975), 201-208.

U-Hang Ki and He-Jin Kim
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea

Byung Hak Kim
Department of Mathematics
Kyung Hee University
Yongin 449-701, Korea