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EXTREME VALUES OF A GAUSSIAN PROCESS

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1. Introduction and Results

Let $\{X(t) : 0 \le t < \infty\}$ be an almost surely continuous Gaussian process with $X(0) = 0$, $E\{X(t)\} = 0$ and stationary increments $E\{X(t)-X(s)\}^2 = \sigma^2(|t-s|)$, where $\sigma(y)$ is a function of $y \ge 0$ (e.g., if $\{X(t); 0 \le t < \infty\}$ is a standard Wiener process, then $\sigma(t) = \sqrt{t}$). Assume that $\sigma(t)$, $t > 0$, is a nondecreasing continuous, regularly varying function at infinity with exponent γ for some $0 < \gamma < 1$. A positive function $\sigma(t)$, $t > 0$, is said to be *regularly varying at infinity with exponent* $\gamma > 0$ if, for all $x > 0$, one has

$$\lim_{t\to\infty} \frac{\sigma(xt)}{\sigma(t)} = x^{\gamma}.$$

Let a_T $(0 < T < \infty)$ be a function of T for which

(i) a_T is nondecreasing,
(ii) $0 < a_T \le T$,
(iii) T/a_T is nondecreasing,

and denote, for $e < T < \infty$,

$$\alpha_T = \left(2\sigma^2(a_T)(2\log T - \log a_T)\right)^{-1/2}$$

and

$$\beta_T = \left(2\sigma^2(a_T)(\log(T/a_T) + \log\log T)\right)^{-1/2}.$$

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Define continuous parameter processes $G_1(T)$, $G_2(T)$, $\cdots$, $G_{10}(T)$ by

$$\begin{aligned}
G_1(T) &= \sup_{0\le s\le a_T}\ \sup_{0\le t\le T-s} |X(t+s)-X(t)|,\\
G_2(T) &= \sup_{0\le s\le a_T}\ \sup_{0\le t\le T-s} \big(X(t+s)-X(t)\big),\\
G_3(T) &= \sup_{0\le s\le a_T}\ \sup_{0\le t\le T-a_T} |X(t+s)-X(t)|,\\
G_4(T) &= \sup_{0\le s\le a_T}\ \sup_{0\le t\le T-a_T} \big(X(t+s)-X(t)\big),\\
G_5(T) &= \sup_{0\le t\le T-a_T} |X(t+a_T)-X(t)|,\\
G_6(T) &= \sup_{0\le t\le T-a_T} \big(X(t+a_T)-X(t)\big),\\
G_7(T) &= \sup_{0\le s\le a_T} |X(T+s)-X(T)|,\\
G_8(T) &= \sup_{0\le s\le a_T} \big(X(T+s)-X(T)\big),\\
G_9(T) &= |X(T+a_T)-X(T)|,\\
G_{10}(T) &= X(T+a_T)-X(T),
\end{aligned}$$

respectively. Clearly, $G_1(T)$ is the largest process and $G_{10}(T)$ is the smallest one of all $G_i(T)$, $i=1,2,\cdots,10$.

Our main object of this paper is to obtain the almost sure limiting values of $G_i(T)$, $i=1,2,\cdots,10$, under the varying conditions on a_T. Thus we are concerned only with behaviors of functions near at infinity. In our proof we shall use the small letter c for a positive constant which may be different from line to line if necessary.

An upper bound for $\alpha_T G_i(T)$ is estimated as follows:

THEOREM 1.1. *Let $\{X(t): 0\le t<\infty\}$ be an almost surely continuous Gaussian process with $X(0)=0, E\{X(t)\}=0$ and $E\{X(t)-X(s)\}^2=\sigma^2(|t-s|)$. Assume that $\sigma(t), t>0$, is a nondecreasing continuous, regularly varying function at ∞ with exponent γ for some $0<\gamma<1$. Let a_T $(0<T<\infty)$ satisfy the conditions such that*

(i) *a_T is nondecreasing,*

(ii) $0<a_T\le T$,

(iii) T/a_T *is nondecreasing*
and
(iv) $\lim_{T\to\infty}(\log T - \log a_T)/\log\log T = r, \qquad 0 \le r \le \infty.$
Then we have

$$\limsup_{T\to\infty} \alpha_T\, G_i(T) \le \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

where $i = 1, 2, \cdots, 10$.

When $r < \infty$, this theorem does not hold if we substitute β_T for α_T, and thus the $\alpha_T$is a critical normalizing factor to get the theorem. It is interesting to compare this theorem with the "limsup" theorems of Csáki et al. [2] and Choi [1].

The next theorem is easily proved by the same way as the proof of Theorem 1.1:

THEOREM 1.2. *Let* $X(t)$ *and* $\sigma(t)$ *be as in Theorem 1.1. Let* a_T $(0 < T < \infty)$ *satisfy the conditions* (i), (ii) *and* (iii) *of Theorem 1.1 and the condition*
(iv)′ $\lim_{T\to\infty}(\log T - \log a_T)/\log T = r, \qquad 0 \le r \le 1.$
Then we have

$$\limsup_{T\to\infty} \alpha_T\, G_i(T) \le \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

where $i = 1, 2, \cdots, 10$.

Sufficient conditions for lower bounds of $\beta_T G_i(T)$ concerning "liminf" are obtained:

THEOREM 1.3. *Let* $\{X(t) : 0 \le t < \infty\}$ *be an almost surely continuous Gaussian process with* $X(0) = 0, E\{X(t)\} = 0$ *and* $E\{X(t) - X(s)\}^2 = \sigma^2(|t-s|)$. *Assume that* $\sigma(t), t > 0$, *is a nondecreasing continuous, regularly varying function at* ∞ *with exponent* γ *for some* $0 < \gamma < 1$. *Let* a_T $(0 < T < \infty)$ *satisfy the conditions such that*
(i) a_T *is nondecreasing,*
(ii) $0 < a_T \le T$,
(iii) T/a_T *is nondecreasing*
and

(iv) $\lim_{T\to\infty}(\log T - \log a_T)/\log\log T = r, \qquad 0 \le r \le \infty.$
Assume that for any $a \le b \le c \le d$
(v) $E\{(X(b)-X(a))(X(d)-X(c))\} \le 0$, *(or,* $\sigma^2(t)$ *is cocave for* $t > 0$*).*
Then we have

$$\liminf_{T\to\infty} \beta_T\, G_i(T) \ge \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

where $i = 1, 2, \cdots, 6$ *if* $r > 0$*; and* $i = 1, 3, 5, 7, 9$ *if* $r = 0$.

THEOREM 1.4. *Let* $X(t)$ *and* $\sigma(t)$ *be as in Thoerem 1.3. Let* a_T $(0 < T < \infty)$ *satisfy the conditions such that*
(i) a_T *is nondecreasing,*
(ii) $0 < a_T \le T$,
(iii) T/a_T *is nondecreasing*
and
(iv)$''$ $\lim_{T\to\infty}(\log T - \log a_T)/\log\log T = r, \qquad 1 < r \le \infty.$
Assume that for $t > 0$,
(v)$'$ $\sigma^2(t)$, $t > 0$, *is twice continuously differentiable which satisfies*

$$|(\sigma^2(t))''| \le c\sigma^2(t)/t^2 \quad \text{for some } c > 0.$$

Then we have

$$\liminf_{T\to\infty} \beta_T\, G_i(T) \ge \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

where $i = 1, 2, \cdots, 6$

For instance, we can choose a_T as 1, $\log T$, T^θ $(0 < \theta < 1)$, $T/(\log T)^r$ $(0 < r < \infty)$ and $cT (0 < c \le 1)$, etc. These theorems show us that we can get exact lower bounds of $\beta_T G_i(T)$ according to the functions a_T to be chosen. Of course, if we replace β_T in Theorems 1.3 and 1.4 by α_T, then they do not hold.

By using the proof process of Theorems 1.3 and 1.4, one can easily obtain the following

THEOREM 1.5. *Let $X(t)$ and $\sigma(t)$ be as in Thoerem 1.3. Let a_T $(0 < T < \infty)$ satisfy the conditions such that*

(i) a_T *is nondecreasing,*

(ii) $0 < a_T \leq T$,

(iii) T/a_T *is nondecreasing*

and

(iv)′ $\lim_{T\to\infty}(\log T - \log a_T)/\log T = r, \qquad 0 \leq r \leq 1.$

Assume that either, for any $a \leq b \leq c \leq d$

(v) $E\{(X(b) - X(a))(X(d) - X(c))\} \leq 0$, (*or,* $\sigma^2(t)$ *is concave for* $t > 0$)

or

(v)′ $\sigma^2(t)$, $t > 0$, is twice continuously differentiable which satisfies

$$|(\sigma^2(t))''| \leq c\sigma^2(t)/t^2 \quad for\ some\ c > 0.$$

Then we have

$$\liminf_{T\to\infty}\ \alpha_T\ G_i(T) \geq \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

where $i = 1, 2, \cdots, 6$ if $0 < r \leq 1$; and $i = 1, 3, 5, 7, 9$ if $r = 0$.

Combining Thoerems 1.2 and 1.5, we have a "limit" value:

THEOREM 1.6. *Let $X(t)$, $\sigma(t)$ and a_T be as in Thoerem 1.5. Further assume that the conditions* (v) *and* (v)′ *of Theorem 1.5 are satisfied. Then we have*

$$\lim_{T\to\infty}\ \alpha_T\ G_i(T) = \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

where $i = 1, 2, \cdots, 6$ if $0 < r \leq 1$; and $i = 1, 3, 5, 7, 9$ if $r = 0$.

If $\sigma(t) = t^\gamma (0 < \gamma < 1)$, then the process $\{X(t); 0 \leq t < \infty\}$ is a fractional Brownian motion of index γ. It is clear that in case of $0 < \gamma \leq 1/2$, the condition (v) of Theorem 1.5 is satisfied, and if $0 < \gamma < 1$ then the condition (v)′ of Theorem 1.5 is satisfied. Hence, Theorem 1.6 can be applied to every fractional Brownian motion.

2. Proofs

When $r = \infty$ in our theorems, let us define $r/(1+r)$ by 1, then the results immediately follow from Csáki et al. [2] and Choi [1]. If $r = 0$ in Theorems 1.3, 1.5 and 1.6, the results are obvious. So we shall prove the theorems when $0 < r < \infty$. The following lemma is essential to prove Theorem 1.1:

LEMMA 2.1. (Choi [1]) *Let $X(t)$ and $\sigma(t)$ be as in Theorem 1.1. Let a_T $(0 < T < \infty)$ satisfy the conditions* (i), (ii) *and* (iii) *of Theorem 1.1. Set*

$$G(s,t) = \frac{X(t+s) - X(t)}{\sigma(a_T)}, \qquad 0 \le s \le a_T,\ 0 \le t \le T - s.$$

Then for any small $\epsilon > 0$ there exist constants $T_0 = T_0(\epsilon)$ and c_ϵ depending only on ϵ such that for all $u \ge 0$ and $T \ge T_0$

$$\mathbf{P}\left\{ \sup_{0 \le s \le a_T} \sup_{0 \le t \le T-s} G(s,t) \ge u \right\} \le c_\epsilon \Big(\frac{T}{a_T}\Big) e^{-u^2/(2+\epsilon)}.$$

Proof of Theorem 1.1. Set $\theta = (r+\epsilon)/(1+r)$ for any small $\epsilon > 0$. Applying Lemma 2.1, we have, from the condition (iv),

$$\begin{aligned}
&\mathbf{P}\left\{ \alpha_T\, G_1(T) > \sqrt{\theta(1+\epsilon)} \right\} \\
&\le 2\mathbf{P}\left\{ \sup_{0 \le s \le a_T} \sup_{0 \le t \le T-s} \frac{X(t+s) - X(t)}{\sigma(a_T)} \right. \\
&\qquad\qquad \left. > \sqrt{2\theta(1+\epsilon)\{\log(T/a_T) + \log T\}} \right\} \\
&\le c_\epsilon \Big(\frac{T}{a_T}\Big) \exp\Big\{ -\frac{1}{2+\epsilon} (2\theta(1+\epsilon)\{\log(T/a_T) + \log T\}) \Big\} \\
&= c_\epsilon \Big(\frac{T}{a_T}\Big) \Big(\frac{T^2}{a_T}\Big)^{-2\theta(1+\epsilon)/(2+\epsilon)} \\
&= c_\epsilon \Big(\frac{T}{a_T}\Big)^{1 - \{(2+2\epsilon)/(2+\epsilon)\}\{(r+\epsilon)/(1+r)\}} T^{-\theta(2+2\epsilon)/(2+\epsilon)} \\
&\le c_\epsilon (\log T)^{(r+\epsilon)\{1 - \{(2+2\epsilon)/(2+\epsilon)\}\{(r+\epsilon)/(1+r)\}\}} T^{-\theta(2+2\epsilon)/(2+\epsilon)} \\
&\le c_\epsilon (\log T)^{\theta} T^{-\theta(1 + \{\epsilon/(2+\epsilon)\})} \\
&\le c_\epsilon T^{-\theta/2}
\end{aligned}$$

provided T is big enough. For given $k \in N$, let $T_k = \exp(k^\alpha)$ where $1/2 < \alpha < 1$. Then the above statement gives

$$\mathbf{P}\left\{\alpha_{T_k} G_1(T_k) > \sqrt{\theta(1+\epsilon)}\right\} \le c_\epsilon e^{-\frac{1}{2}\theta k^\alpha}.$$

The series

$$\sum_k \mathbf{P}\left\{\alpha_{T_k} G_1(T_k) > \sqrt{\theta(1+\epsilon)}\right\}$$

is convergent, and using the Borel-Cantelli lemma, we have

$$\limsup_{k\to\infty} \ \alpha_{T_k} \ G_1(T_k) \le \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

The remainder of the proof is to show that

$$\limsup_{T\to\infty} \ \alpha_T \ G_1(T) \le \limsup_{k\to\infty} \ \alpha_{T_k} \ G_1(T_k). \tag{2.1}$$

Let T be in $T_{k-1} \le T \le T_k$. Then, by the conditions (i) $\sim$ (iii),

$$\alpha_{T_k} G_1(T_k) \ge \sup_{0\le s\le a_T} \sup_{0\le t\le T-s} \frac{|X(t+s) - X(t)|}{\sqrt{2(2\log T_k - \log a_{T_k})}\sigma(a_{T_k})}$$

$$\ge \alpha_T G_1(T) \left\{\frac{2\log T_{k-1} - \log a_{T_k-1}}{2\log T_k - \log a_{T_k}}\right\}^{1/2} \frac{\sigma(a_{T_{k-1}})}{\sigma(a_{T_k})}. \tag{2.2}$$

From the conditions (i) $\sim$ (iii) and using the fact that $(\log u)/u$ is decreasing for $u > e$, we have

$$1 \ge \frac{2\log T_{k-1} - \log a_{T_{k-1}}}{2\log T_k - \log a_{T_k}} \ge \left(\frac{T_{k-1}}{T_k}\right)^2 \frac{a_{T_k}}{a_{T_{k-1}}}$$

$$\ge \left(\frac{T_{k-1}}{T_k}\right)^2 = \exp\{2((k-1)^\alpha - k^\alpha)\}$$

$$\ge \exp\{-2\alpha(k-1)^{\alpha-1}\} \to 1 \qquad \text{as } k\to\infty. \tag{2.3}$$

and

$$1 \ge \frac{a_{T_{k-1}}}{a_{T_k}} \ge \frac{T_{k-1}}{T_k} \ge \exp\{-\alpha(k-1)^{\alpha-1}\}.$$

Thus it follows from the regularity of $\sigma(\cdot)$ at ∞ that

$$
\begin{aligned}
1 \geq \frac{\sigma(a_{T_{k-1}})}{\sigma(a_{T_k})} &\geq \frac{\sigma\big(\exp\{-\alpha(k-1)^{\alpha-1}\}a_{T_k}\big)}{\sigma(a_{T_k})} \\
&\geq \exp\big\{-\alpha(k-1)^{\alpha-1}(\gamma+\eta)\big\} \to 1 \qquad \text{as } k\to\infty, \qquad (2.4)
\end{aligned}
$$

where $\eta > 0$ is small enough. Combining (2.2), (2.3) and (2.4), we obtain the inequality (2.1). This completes the proof of Theorem 1.1.

For proving Theorem 1.3, we need the following lemmas:

LEMMA 2.2. (Slepian [4]) *Let $G(t)$ and $G^*(t)$, $0 \leq t < \infty$, be centered Gaussian processes, possessing continuous sample path functions, with $E\{G(t)\}^2 = E\{G^*(t)\}^2 = 1$, and let $\rho(s,t)$ and $\rho^*(s,t)$ be their respective covariance functions. Suppose that we have*

$$\rho(s,t) \geq \rho^*(s,t), \qquad 0 \leq s,t < \infty.$$

Then

$$\mathbf{P}\Big\{\sup_{0\leq t\leq T} G(t) \leq u\Big\} \geq \mathbf{P}\Big\{\sup_{0\leq t\leq T} G^*(t) \leq u\Big\}.$$

LEMMA 2.3. (Choi [1]) *Let $X(t)$, $\sigma(t)$ and a_T be as in Theorem 1.3. For $0 < \alpha < 1$ set $T_k = \exp(k^\alpha)$, $k \in N$, and let T be in $T_k \leq T \leq T_{k+1}$. Then we have*

$$\liminf_{T\to\infty} \beta_T\, G_6(T) \geq \liminf_{k\to\infty} \beta_{T_k}\, G_6(T_k) \qquad \text{a.s.}$$

Proof of Theorem 1.3. For given T large, let us define a positive integer h_T by $h_T = [T/a_T]$, where $[x]$ denotes the greatest integer not exceeding x. It is clear from the conditions (iv) that h_T is increasing. For $i = 1, 2, \cdots, h_T$, we define the incremental random variable

$$Z_T(i) = X(ia_T) - X((i-1)a_T).$$

Clearly $Z_T(i)/\sigma(a_T)$ is a standard normal random variable. By the condition (v), we have

$$covariance(Z_T(i), Z_T(j)) \leq 0, \quad i \neq j.$$

For any small $\epsilon > 0$, we set

$$b = \frac{r - 2\epsilon}{(1 - (\epsilon/2))(1 + r - \epsilon)} > 0.$$

Applying Lemma 2.2 for $G^*(i) = Z_T(i)/\sigma(a_T)$, $i = 1, 2, \cdots, h_T$, we have

$$\begin{aligned}
&\mathbf{P}\big\{\beta_T\, G_6(T) < \sqrt{b(1-\epsilon)}\big\} \\
&\le \mathbf{P}\Big\{\sup_{1 \le j \le h_T} \frac{Z_T(j)}{\sigma(a_T)} < \sqrt{2b(1-\epsilon)(\log(T/a_T) + \log\log T)}\Big\} \\
&\le \big\{\Phi(u_T)\big\}^{h_T}
\end{aligned}$$

where $u_T = \sqrt{2b(1-\epsilon)(\log(T/a_T) + \log\log T)}$ and $\Phi(\cdot)$ denotes the standard normal distribution function. Since, for large T,

$$\Phi(u_T) = 1 - \mathbf{P}(Z \ge u_T) \le e^{-\mathbf{P}(Z \ge u_T)}$$

and for some $c > 0$

$$\begin{aligned}
\mathbf{P}(Z \ge u_T) &\ge c\exp(-\frac{1}{2}u_T^2) \\
&= c\Big(\frac{T}{a_T}\log T\Big)^{-b(1-\epsilon)},
\end{aligned}$$

we have

$$\begin{aligned}
\big\{\Phi(u_T)\big\}^{h_T} &\le \exp\Big\{-c\Big(\frac{T}{a_T}\Big)\Big(\frac{T}{a_T}\log T\Big)^{-b(1-\epsilon)}\Big\} \\
&\le \exp\Big\{-c\Big(\frac{T}{a_T}\Big)^{1-b(1-(\epsilon/2))}(\log T)^{-b(1-(\epsilon/2))}\Big\}.
\end{aligned}$$

Using the condition (iv), we get

$$\frac{T}{a_T} \ge (\log T)^{r-\epsilon}$$

and

$$\begin{aligned}
\big\{\Phi(u_T)\big\}^{h_T} &\le \exp\big\{-c(\log T)^{(r-\epsilon)\{1-b(1-(\epsilon/2))\}-b(1-(\epsilon/2))}\big\} \\
&= \exp\big\{-c(\log T)^{\epsilon}\big\}.
\end{aligned}$$

Therefore we obtain, for all large T,

$$\mathbf{P}\{\beta_T G_6(T) < \sqrt{b(1-\epsilon)}\} \le \exp\{-c(\log T)^{\epsilon}\}.$$

For $0 < \alpha < 1$, set $T_k = \exp(k^{\alpha}), k \in N$. Then

$$\mathbf{P}\{\beta_{T_k} G_6(T_k) < \sqrt{b(1-\epsilon)}\} \le \exp(-ck^{\alpha\epsilon})$$

and the series

$$\sum_k \mathbf{P}\{\beta_{T_k} G_6(T_k) < \sqrt{b(1-\epsilon)}\}$$

is convergent. The Borel-Cantelli lemma implies

$$\liminf_{k\to\infty} \beta_{T_k} G_6(T_k) \ge \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

Let T be in $T_k \le T \le T_{k+1}$ for given T_k. Then Lemmma 2.3 completes the proof of Theorem 1.3.

The next lemmas are applied to prove Theorem 1.4:

LEMMA 2.4. (Leadbetter et al. [3]) *Let* $\{X_i; i = 1, 2, \cdots, n\}$ *be jointly standardized normal random variables with* covariance (X_i, X_j) $=\Lambda_{ij}$ *such that*

$$\delta = \max_{i\neq j} |\Lambda_{ij}| < 1.$$

Then for any real number u *and integers* $1 \le l_1 < l_2 < \cdots < l_k \le n$ *with* $k \le n$,

$$P\left\{\max_{1\le j\le k} X_{l_j} \le u\right\} \le \{\Phi(u)\}^k$$

$$+ K \sum_{1\le i<j\le k} |r_{ij}| \exp\Big(-\frac{u^2}{1+|r_{ij}|}\Big) \tag{2.5}$$

where $r_{ij} = \Lambda_{l_i l_j}$ *and* $K = K(\delta)$ *is a positive constant depending on* δ *but not* n, u *and* k, *and* $\Phi(\cdot)$ *denotes the standard normal distribution function.*

We shall estimate an upper bound of the second term in the above inequality (2.5) by imposing a stationary condition concerning covariance functions of $\{X_i; i = 1, 2, \cdots, n\}$. Note that in the condition (iv)$''$ of Theorem 1.4, when $T > 0$ is a large number, one can choose a big integer $M > 0$ such that $M < (\log T)^B < T/a_T$ for some $B > 0$. The following lemma is easily verified by the same way as the proof of Lemma 4.4(ii) in Choi [1]:

LEMMA 2.5. *Let $X_i (i = 1, 2, \cdots, n)$, δ and r_{ij} be given as in Lemma 2.4. Assume that the covariance functions r_{ij} be such that*

$$|r_{ij}| \leq \rho_{|i-j|} < 1 \quad \text{for} \quad i \neq j.$$

Suppose that the function $a_T (0 < T < \infty)$ is as in Theorem 1.4. For given $T > 0$ large, set $k = [T/(Ma_T)]$, where $[x]$ denotes the greatest integer not exceeding x, and let, for some $\nu > 0$,

$$\rho_m < m^{-\nu} \quad \text{for all} \quad m = |i-j| = 1, 2, \cdots, k-1.$$

For $0 < r < \infty$, let $u = \sqrt{2b(1-\epsilon)\{\log(T/a_T) + \log\log T\}}$ and $b = (r - 2\epsilon)/\{(1 - (\epsilon/2))(1 + r - \epsilon)\} > 0$ for any small $\epsilon > 0$. Then there exists $K > 0$ depending only on ϵ, δ and ν such that

$$\sum_{1 \leq i < j \leq k} |r_{ij}| \exp\Big(-\frac{u^2}{1 + |r_{ij}|}\Big) \leq K(\log T)^{-\delta_0}$$

where $\delta_0 = \{\{r\nu(1+r)(1-\delta)\}/\{(1+\nu)(1+\delta)(1-(\epsilon/2))(1+r-\epsilon)\}\} - \epsilon'$ and $\epsilon' > 0$ is small enough.

Proof of Theorem 1.4. Let $1 < r < \infty$ in the condition (iv)$''$. Then, for $T > 0$ large, we can choose a big integer $M > 0$ such that $M < (\log T)^B < T/a_T$ for some $B > 0$. Define a positive integer k_T by $k_T = [T/(Ma_T)]$ as in Lemma 2.5. By (iv)$''$, k_T is increasing. For $i = 1, 2, \cdots, k_T$, we define the incremental random variable

$$Y_T(i) = X(Mia_T) - X((Mi-1)a_T).$$

Then $Y_T(i)/\sigma(a_T)$ is a standard normal random variable. It follows that for large $T > 0$,

$$\begin{aligned} &\mathbf{P}\{\beta_T G_6(T) < \sqrt{b(1-\epsilon)}\} \\ (2.6) \qquad &\leq \mathbf{P}\Big\{\sup_{1 \leq i \leq k_T} \frac{Y_T(i)}{\sigma(a_T)} < \sqrt{2b(1-\epsilon)\{\log(T/a_T) + \log\log T\}}\Big\}. \end{aligned}$$

Let $r_T(i,j) = \mathit{correlation}(Y_T(i), Y_T(j))$, $i \neq j$, and let $m = |i - j| \geq 1$. By the same process as the proof of Theorem 2.2 in Choi [1] (Here we make use of the condition (v)′), we can get $|r_T(i,j)| < m^{-\nu}$ where $\nu = 1-\gamma > 0$. Applying Lemmas 2.4 and 2.5 for $X_{l_j} = Y_T(j)/\sigma(a_T)$, $j = 1, 2, \cdots, k_T$, and $u_T = \sqrt{2b(1-\epsilon)\{\log(T/a_T) + \log\log T\}}$, the last term of (2.6) is less than or equal to

$$\{\Phi(u_T)\}^{k_T} + K(\log T)^{-\delta_0}$$

where $\delta_0 = \{\{r\nu(1+r)(1-\delta)\}/\{(1+\nu)(1+\delta)(1-(\epsilon/2))(1+r-\epsilon)\}\} - \epsilon'$ and $\epsilon' > 0$ is small enough. Thus we have

$$\mathbf{P}\{\beta_T G_6(T) < \sqrt{b(1-\epsilon)}\} \le \exp\{-c(\log T)^{\epsilon}\} + K(\log T)^{-\delta_0}.$$

This inequality is bounded by $K(\log T)^{-\delta_0}$. For given $k \in N$, let us set $T_k = \exp(k^{\alpha})$, where α is taken by

$$1 > \alpha = \frac{(1+\nu)(1+\delta)(1-(\epsilon/2))(1+r-\epsilon)}{r\nu(1+r)(1-\delta)} + \epsilon' > 0.$$

Then we have

$$\mathbf{P}\left\{\beta_{T_k} G_6(T_k) < \sqrt{b(1-\epsilon)}\right\} \le K k^{-\alpha\delta_0}$$

and the series

$$\sum_k \mathbf{P}\left\{\beta_{T_k} G_6(T_k) < \sqrt{b(1-\epsilon)}\right\}$$

is convergent, and hence the Borel-Cantelli lemma implies

$$\liminf_{k\to\infty} \ \beta_{T_k} G_6(T_k) \ge \sqrt{\frac{r}{1+r}} \qquad \text{a.s.}$$

Letting T be in $T_k \le T \le T_{k+1}$ for given T_k, Theorem 1.4 immediately follows from Lemma 2.3.

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