J. Korean Math. Soc. **32** (1995), No. 4, pp. 713–724

GENERATING FUNCTION OF CELLS OF GENERALIZED YOUNG TABLEAUX

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1. Introduction

In 1954 Frame, Robinson and Thrall [5] gave the hook formula for the number of standard Young tableaux of a given shape. Since then many proofs for the hook formula have been given using various methods. See [9] for probabilistic method and see [6] or [12] for combinatorial ones. Regev [10] has given asymptotic values for these numbers and Gouyou–Beauchamps [8] gave exact formulas for the number of standard Young tableaux having n cells and at most k rows in the cases $k = 4$ and $k = 5$.

Recently, the generating function of weights of generalized Young tableaux with special conditions has been studied. Gordon [7] has given the generating function of weights of such tableaux having at most m rows and Désarménien [4] has also given the generating function of weights of the generalized Young tableaux with height bounded by $2k + 1$ and p columns having an odd number of elements.

In this paper, we give the generating function for the number of cells of the generalized Young tableaux with bounded by height $2k$. Proofs are purely combinatorial. We give a bijection between the set of configurations of disjoint paths and the set of the generalized Young tableaux. These configurations of paths will be enumerated by a Pfaffian, which has been introduced by Desainte-Catherine [2].

2. Pfaffian

Let $\mathbf{Z}$ be the set of all integers and let $\Pi = \mathbf{Z} \times \mathbf{Z}$. A path of Π is a sequence $w = (s_0, s_1, \cdots, s_n)$ of points in Π such that if $s_i = (x, y)$, then s_{i+1} is either $(x, y+1)$ (an East step) or $(x-1, y)$ (a North step).

Received September 15, 1994.
1991 AMS Subject Classification: 05A15.
Key words: Pfaffian, nonintersecting path, generalized Young tableaux.

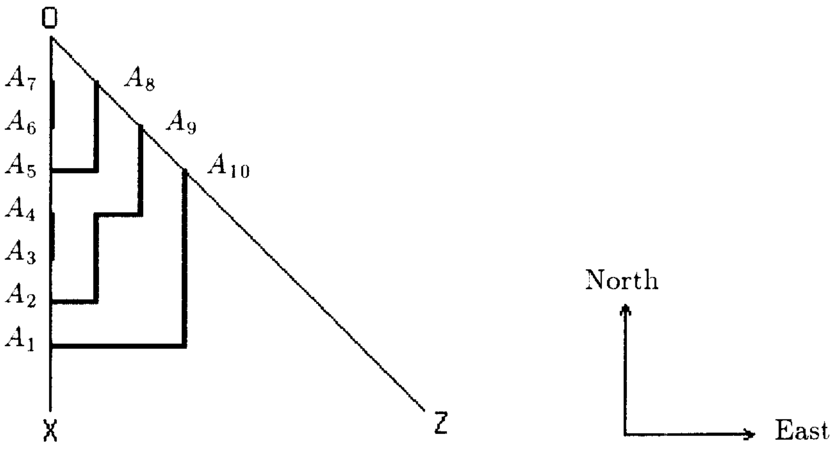

Figure 2.1: Configuration of nonintersecting path for $m = 7$ and $n = 3$

We say that s_0 is the *starting point* and s_n is the *arriving point* of w. See Figure 2.1.

We consider two half lines OX and OZ having lines equations $y = 0$ and $y = x$, respectively. Let $A_1, A_2, \ldots, A_m, A_{m+1}, \ldots, A_{m+n}$ be points in Π with the following coordinates:

$$A_i = \begin{cases} (m+1-i, 0) & if \quad 1 \leq i \leq m \\ (i-m, i-m) & if \quad m+1 \leq i \leq m+n. \end{cases}$$

Note that $A_1, A_2, \cdots, A_m$ are consecutive points on OX and A_{m+1}, $A_{m+2}, \cdots, A_{m+n}$ are consecutive points on OZ. In this paper we assume that $m+n$ is even.

For any path w going from A_i to A_j, we give weights for elementary steps (East step or North step) contained in w and we define the weight $v(w)$ of the path w as product of weights of elementary steps in w. Then we can define a matrix $T = (t_{i,j})_{1 \leq i,j \leq m+n}$ such that $t_{i,j} = \sum_{w: A_i \to A_j} v(w)$, where the summation is over all the paths going from A_i to A_j.

Note that $t_{i,j} = 0$, if there is no path going from A_i to A_j.

DEFINITION 2.1. Let S_{2k} be the symmetric group of degree $2k$. An element $\alpha \in S_{2k}$ is called an *involution* if α^2 is the identity in S_{2k}. An involution $\alpha \in S_{2k}$ is *without crossing* if there is no pair (i,j) such that $1 \leq i < j < \alpha(i) < \alpha(j) \leq 2k$.

DEFINITION 2.2. A k-tuple of path $(w_1, w_2, ..., w_k)$ is *compatible* with an involution $\alpha \in S_{2k}$ without fixed point if, for $1 \leq i \leq k$, w_i links $A_{\tau(i)}$ to $A_{\alpha(\tau(i))}$, where τ is the map from $[1,k]$ to $[1,2k]$ defined by

(1) $\tau(1) < \tau(2) < ... < \tau(k)$ and
(2) $\tau(i) < \alpha(\tau(i))$ for $1 \leq j \leq k$.

That is, τ is the numbering of pairs $(i, \alpha(i))$ in the order which we meet them, when we run through the integers from 1 to $2k$.

EXAMPLE. If $k = 5$ and $\alpha = (10, 9, 4, 3, 8, 7, 6, 5, 2, 1)$, then $\tau(1) = 1$, $\tau(2) = 2$, $\tau(3) = 3$, $\tau(4) = 5$, $\tau(5) = 6$.

THEOREM 2.3 ([2]). *Let $2k = m + n$. Then the Pfaffian $Pf(T)$ associated with the matrix T is equal to*

$$Pf(T) = \sum_{(\alpha, w_1, ..., w_k)} v(w_1)v(w_2)...v(w_k),$$

where the summation is over all the $(k+1)$-tuples $(\alpha, w_1, ..., w_k)$ such that α is an involution of length $2k$ without crossing and without fixed point, and such that $(w_1, w_2, ..., w_k)$ is a k-tuple of disjoint paths, compatible with the involution α.

THEOREM 2.4 (CAYLEY 1847). *Let $T = (t_{i,j})$ be a matrix such that, for $1 \leq i < j \leq 2k$, $t_{i,j} = 0$, and let M be an anti-symmetric matrix $M = T - T^t$ (where T^t denote the transpose of T). If we denote $Pf(T)$ the Pfaffian of the matrix T, then*

$$Det(M) = (Pf(T))^2.$$

DEFINITION 2.5. A *partition* λ is a sequence $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ of nonnegative integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. The *Ferrers diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ is an array of m rows of cells, with

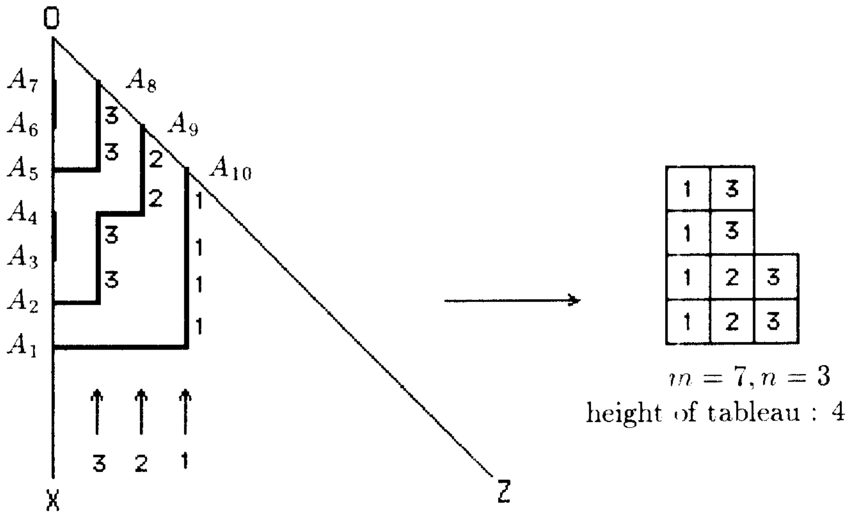

Figure 2.2: Configuration of nonintersecting path and generalized Young tableau with columns having an even number of elements

λ_i cells, left justified, in the ith row. (Zero parts are ignored.) The rows are numbered from bottom to top.

DEFINITION 2.6. A *generalized Young Tableau* of shape λ is a filling of the Ferrers diagram of λ with positive integers which are increasing from bottom to top and strictly increasing from left to right.

Desainte-Catherine and Viennot [3] set up a bijection between the set of generalized Young tableaux with columns having only an even number of elements, and the k-tuples of paths being disjoint each other, in the following way : for a path, we give the number i to the North steps of path of y-coordinate $n-i+1$; we construct a column of generalized Young tableaux associated in taking all numbers with decreasing order on the path, except the dominos on the axis OX. See Figure 2.2. Note that the objects constructed from this bijection are generalized Young tableaux with height bounded by $m-n$ and columns having only an even number of elements, and with the entries between 1 and n.

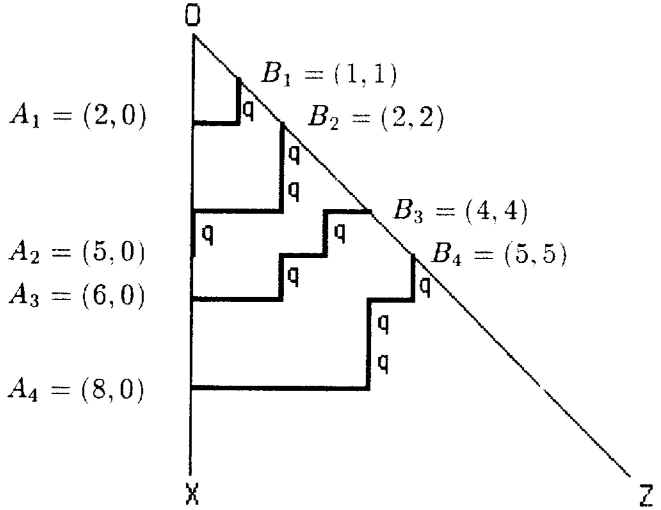

Figure 2.3: Weight of the configuration : q^9

We define a weight on a path going from A_i to A_j in the following way :

(1) a North step of the path is weighted by q,
(2) an East step is weighted by 1.

If we remember that the configurations of paths without a fixed point, being disjoint each other, in this eighth of plane delimited by the axis OX and OZ, correspond to the generalized Young tableaux with columns having only an even number of elements, this weight q of North step counts the cells of such tableaux. See Figure 2.3.

Let $T = (t_{i,j})_{1\leq i,j\leq m+n}$ be a matrix with $t_{i,j} = \sum_{w:A_i\to A_j} v(w)$, where $v(w)$ is the weight of a path w. Then

$$t_{i,j} = \begin{cases} 0 & if\ 1 \leq j \leq i \leq m+n \\ q^{j-i} & if\ 1 \leq i < j \leq m \\ q^{2m-i-j+1}\binom{m-i+1}{2m-i-j+1} & if\ 1 \leq i \leq 2m-j+1\ and \\ & \quad m+1 \leq j \leq m+n \\ 0 & otherwise. \end{cases}$$

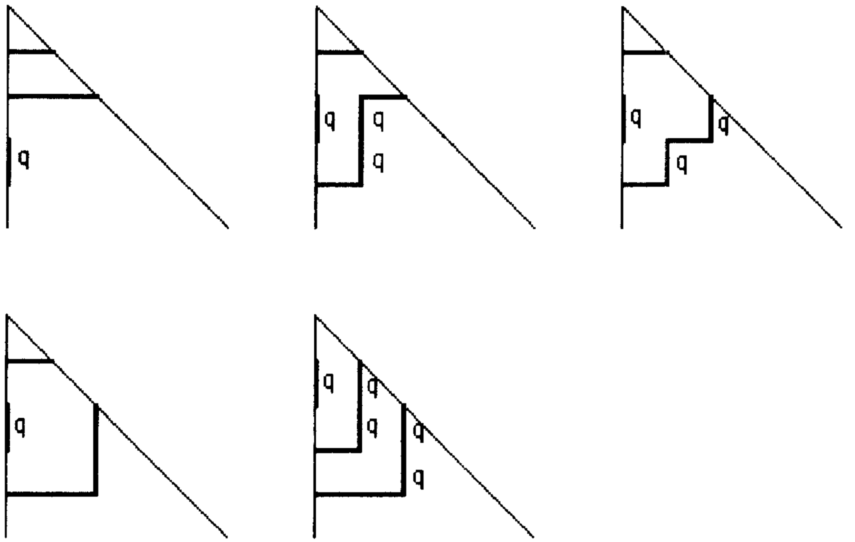

Figure 2.4: Configuration of nonintersecting path for $m = 4$ and $n = 2$

Let M be an anti-symmetric matrix $[m+n] \times [m+n]$ defined by $m_{i,j} = t_{i,j}$ for $1 \le i < j \le m+n$, and $m_{j,i} = -t_{i,j}$ for $1 \le j \le i \le m+n$. According to the Theorem 2.4, $\mathrm{Pf}(T) = (\det(M))^{\frac{1}{2}}$. We denote $Pfc_{m,n}(q)$ the Pfaffian $\mathrm{Pf}(T)$ of the matrix T. For example, we have $Pfc_{4,2}(q) = q(1+3q^2+q^4)$. See Figure 2.4. From Desainte-Catherine and Viennot's bijection and the Theorem 2.3, we know that the Pfaffian of the matrix T counts the number of cells of the generalized Young tableaux with height bounded by $m-n$ and columns having only an even number of elements, having the entries between 1 and n.

EXAMPLE. We have the following anti-symmetric matrix $[m+n] \times [m+n]$ for m=3 and $n = 1$:

$$M = \begin{pmatrix} 0 & q & q^2 & q^2\binom{3}{2} \\ -q & 0 & q & q\binom{2}{1} \\ -q^2 & -q & 0 & 1 \\ -q^2\binom{3}{2} & -q\binom{2}{1} & -1 & 0 \end{pmatrix}$$

We denote $c_{n,m-n}(q) = \sum_i c_{n,m-n,i} q^i$ the generating function of generalized Young tableaux with columns having only an even number of elements, where $c_{n,m-n,i}$ is the number of generalized Young tableaux with height bounded by $m-n$ and columns having only an even number of elements, having i cells, and the entries between 1 and n. Note that $\sum_i c_{n,m-n,i}$ is the number of all generalized Young tableaux with height bounded by $m-n$ and columns having only an even number of elements, and the entries between 1 and n.

Note that

$$\begin{aligned} c_{n,m-n}(q) &= q^{-(m-n/2)} Pf c_{m,n}(q) \\ &= (q^{-(m-n)} det(M))^{\frac{1}{2}}. \end{aligned}$$

In fact, we have $(m-n)/2$ dominos in all of the configurations of paths without fixed point, being disjoint each other in the eighth of plane delimited by the axis OX and OZ, and these dominos play no role in the enumeration of the generalized Young tableaux. For example, we have $c_{2,2}(q) = (1 + 3q^2 + q^4)$ according to the Figure 2.4.

Generally we can not simplify the determinant of the anti-symmetric matrix M. In Section 3, we calculate the extracted minor of the matrix whose coefficients are the products of a monomial by a binomial number.

3. Path and q-determinant

Consider the infinite matrix B $= (b_{i,j})_{i,j \geq 0}$ with $b_{i,j} = q^{i-j} \binom{i}{j}$, where

$$\binom{i}{j} = \begin{cases} \frac{i!}{j!(i-j)!} & if\ i \geq j \\ 0 & otherwise. \end{cases}$$

Hence,

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ q\binom{1}{0} & \binom{1}{1} & 0 & 0 & 0 & \cdots \\ q^2\binom{2}{0} & q\binom{2}{1} & \binom{2}{2} & 0 & 0 & \cdots \\ q^3\binom{3}{0} & q^2\binom{3}{1} & q\binom{3}{2} & \binom{3}{3} & 0 & \cdots \\ q^4\binom{4}{0} & q^3\binom{4}{1} & q^2\binom{4}{2} & q\binom{4}{3} & \binom{4}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

For $0 \leq g_1 < g_2 < \cdots < g_k$ and $0 \leq h_1 < h_2 < \cdots < h_k$, we denote $\begin{pmatrix} g_1 & \cdots & g_k \\ h_1 & \cdots & h_k \end{pmatrix}_{B,q}$ the minor of B with $g_1, \ldots, g_k$ the row indices and $h_1, \ldots, h_k$ the column indices.

Let $G_1, G_2, \ldots, G_k$ and $H_1, H_2, \ldots, H_k$ be points of the plain Π, such that $G_i = (g_i, 0)$ and $H_i = (h_i, h_i), 1 \leq i \leq k$.

We define a weight on a path going from G_i to H_i, $1 \leq i \leq k$, in Π, in the following way :

(1) a North step is weighted by q.
(2) an East step is weighted by 1.

According to Theorem 2.3, we know that the determinant $\begin{pmatrix} g_1 & \cdots & g_k \\ h_1 & \cdots & h_k \end{pmatrix}_{B,q}$ is equal to the sum of weight of configurations of k-tuples $(w_1, w_2, \ldots, w_k)$ of paths of Π such that

(i) w_i go from G_i to H_i for $1 \leq i \leq k$, ;
(ii) the paths are disjoint each other.

In this section we evaluate the determinant $\begin{pmatrix} g_1 & \cdots & g_k \\ h_1 & \cdots & h_k \end{pmatrix}_{B,q}$ and obtain the closed form for the generating function of the number of cells of the generalized Young tableaux bounded by height $2k$.

LEMMA 3.1.

$$\begin{pmatrix} g_1 & \cdots & g_k \\ h_1 & \cdots & h_k \end{pmatrix}_{B,q} = \frac{g_1 \cdots g_k}{h_1 \cdots h_k} \begin{pmatrix} g_1 - 1 & \cdots & g_k - 1 \\ h_1 - 1 & \cdots & h_k - 1 \end{pmatrix}_{B,q}.$$

Proof. We can obtain this result by the following calculation :

$$\begin{aligned} \binom{g}{h}_{B,q} &= q^{g-h}\binom{g}{h} = q^{g-h}\frac{g}{h}\binom{g-1}{h-1} \\ &= \frac{g}{h}\binom{g-1}{h-1}_{B,q}. \end{aligned}$$

□

LEMMA 3.2. *We have the following equality :*

$$\begin{pmatrix} g & g+1 & \cdots & g+k-1 \\ 0 & h_2 & \cdots & h_k \end{pmatrix}_{B,q} = q^g \begin{pmatrix} g & \cdots & g+k-2 \\ h_2-1 & \cdots & h_k-1 \end{pmatrix}_{B,q}.$$

proof. The determinant $\begin{pmatrix} g & g+1 & \cdots & g+k-1 \\ 0 & h_2 & \cdots & h_k \end{pmatrix}_{B,q}$ is equal to the sum of weights of configurations of k disjoint paths going from $G_i = (g+i-1, 0)$ to $H_i = (h_i, h_i), 1 \le i \le k$ with $h_1 = 0$. Note that the path going from G_1 to H_1 consists uniquely of North step and his valuation is q^g. Furthermore, since the starting points of paths are consecutive in all these configurations, all the first steps going from G_i to $H_i, 2 \le i \le k$ are East steps. If we take away the path from G_1 to H_1 and all the first steps of $k-1$ other paths, then we obtain the new configurations of k disjoint paths such that the sum of weights of such configurations is $q^g \begin{pmatrix} g & g+1 & \cdots & g+k-2 \\ h_2-1 & h_3-1 & \cdots & h_k-1 \end{pmatrix}_{B,q}$. □

LEMMA 3.3.

$$\begin{pmatrix} g_1 & g_2 & \cdots & g_k \\ 0 & 1 & \cdots & k-1 \end{pmatrix}_{B,q} = q^{g_1 k} \begin{pmatrix} g_2-g_1 & \cdots & g_k-g_1 \\ 1 & \cdots & k-1 \end{pmatrix}_{B,q}.$$

Proof. $\begin{pmatrix} g_1 & g_2 & \cdots & g_k \\ 0 & 1 & \cdots & k-1 \end{pmatrix}_{B,q}$ is equal to the sum of weights of configurations of k disjoint paths going from $G_i = (a_i, 0)$ to $H_i = (i-1, i-1), 1 \le i \le k$. Since $H_1, H_2, \ldots, H_k$ are consecutive points on OZ, the final g_1 steps of each path must be North. Remove the path from G_1 to H_1 and the final g_1 North step of each other path. □

From the Lemma 3.2 and Lemma 3.3, we obtain the following theorem:

THEOREM 3.4.

$$\begin{pmatrix} g_1 & \cdots & g_k \\ h & \cdots & h+k-1 \end{pmatrix}_{B,q} = \prod_{i=1}^{k} \frac{(g_i)_h}{(h+i-1)!} \Delta(g_1, \ldots, g_k) q^{\ell},$$

where $(g)_h = g(g-1)\cdots(g-h+1)$, $\ell = g_1 + g_2 + \cdots + g_n - hk - k(k-1)/2$ and where $\Delta(x_1, \ldots, x_k)$ is the Vandermonde's determinant $\prod_{1\le i<j\le n}(x_j - x_i)$.

Proof. Applying the Lemma 3.1 repeatedly, we have

$$\begin{pmatrix} g_1 & \cdots & g_k \\ h & \cdots & h+k-1 \end{pmatrix}_{B,q} = \prod_{i=1}^{k} \frac{(g_i)_h}{(h+i-1)!/(i-1)!} \begin{pmatrix} g_1 - h & \cdots & g_k - h \\ 0 & \cdots & k-1 \end{pmatrix}_{B,q}.$$

From the Lemma 3.3, we also obtain

$$\begin{pmatrix} g_1 - h & \cdots & g_k - h \\ 0 & \cdots & k-1 \end{pmatrix}_{B,q} = q^{(g_1-h)k} \begin{pmatrix} g_2 - g_1 & \cdots & g_k - g_{k-1} \\ 1 & \cdots & k-1 \end{pmatrix}_{B,q}.$$

Applying the Lemma 3.1 again, we get

$$\begin{pmatrix} g_2 - g_1 & \cdots & g_k - g_{k-1} \\ 1 & \cdots & k-1 \end{pmatrix}_{B,q} = \frac{\prod_{i=1}^{k-1}(g_{k+1} - g_k)}{(k-1)!} \begin{pmatrix} g_2 - g_1 - 1 & \cdots & g_k - g_{k-1} - 1 \\ 0 & \cdots & k-2 \end{pmatrix}_{B,q}.$$

If we use Lemma 3.1 and Lemma 3.3, we obtain the following identities

$$\begin{aligned}\begin{pmatrix} g_1 & \cdots & g_k \\ h & \cdots & h+k-1 \end{pmatrix}_{B,q} &= q^{(g_1-h)k} \prod_{2\le i\le k} q^{(g_i - g_{i-1} - 1)(k-i+1)} \\ &\quad \frac{(g_1)_h \cdots (g_k)_h}{h! \cdots (h+k-1)!} \Delta(g_1, \ldots, g_k) \\ &= \prod_{i=1}^{k} \frac{(g_i)_h}{(h+i-1)!} \Delta(g_1, \ldots, g_k) q^{\ell},\end{aligned}$$

as we required.

Let $A_1, \ldots, A_m, A_{m+1}, \ldots, A_{m+n}$ be the points as in Section 2. Recall that $m+n$ is even and $m \ge n$. Consider a configuration without fixed point of paths being disjoint each other in the eighth of plain delimited by the axis OX and OZ. This configuration can contain n disjoint paths $w_1, \ldots, w_n$ which are not dominos. For $i = 1, 2, \ldots, n$ let $D_i = (a_i, 0)$ and $E_i = (i, i)$ be the starting point and the arriving point of w_i respectively, where $a_1 < a_2 < \ldots < a_n$. Note that, for $i = 1, \ldots, n$, D_i's are consecutive or separated by dominos and E_i's are consecutive points on the axis OZ. In this configuration, the first steps of all the paths w_i's must be East. If we take off all these first steps of w_i's, and if we use new axes OX' and OY', we obtain a new configuration. Here, OX' (resp. OY') is obtained from OX (resp. OY) by translating into an East (resp. South) step. In a new configuration, we have new points $D'_i = (a'_i, 0)$ with $a'_i = a_i - 1$ and $E'_i = (i-1, i-1), 1 \leq i \leq n$. Hence we get the following formula.

PROPOSITION 3.5.

$$Pfc_{m,n,q} = q^{(m-n)/2} \sum_{a_1, a_2, \ldots, a_n} \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ 0 & 1 & \ldots & n-1 \end{pmatrix}_{B,q},$$

where the summation is over the sets of n positive integer $\{a_1, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$, such that a_1 is even. for $1 \leq i \leq n$, $a_{i+1} - a_i$ is odd, and $a_i - i - 1 \leq m - n$.

Putting together the Lemma 3.4 and Propositin 3.5, we obtain the following Theorem :

THEOREM 3.6.

$$c_{n,m-n}(q) = \sum_{a_1, a_2, \ldots, a_n} \frac{\Delta(a_1, \ldots, a_n)}{1! \cdots (n-1)!} q^{a + \cdots + a_n - k(k-1)/2},$$

where $\Delta(x_1, \ldots, x_n)$ is the Vandermonde's determinant and where the summation is over the sets of n positive integers $\{a_1, \ldots, a_n\}$ with $a_1 < a_2 < \ldots < a_n$, such that a_1 is even, for $1 \leq i \leq n$, $a_{i+1} - a_i$ is odd, and $a_i - i - 1 \leq m - n$.

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