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A GEOMETRIC CRITERION FOR THE ELEMENT OF THE CLASS $\mathbb{A}^2_{1,\aleph_0}(r)$

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1. Introduction

Let $\mathcal{H}$ denote a separable, infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1_{\mathcal{H}}$ and is closed in the weak* operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains T and $1_{\mathcal{H}}$ and is closed in the weak* operator topology. Moreover, let $Q_{\mathcal{A}_T}$ denote the quotient space $\mathcal{C}_1(\mathcal{H})/\perp_{\mathcal{A}_T}$, where $\mathcal{C}_1(\mathcal{H})$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and $\perp_{\mathcal{A}_T}$ denotes the preannihilator of $\mathcal{A}_T$ in $\mathcal{C}_1(\mathcal{H})$. For a brief notation, we shall denote $Q_{\mathcal{A}_T}$ by Q_T. One knows that $\mathcal{A}_T$ is the dual space of Q_T and that the duality is given by

$$(1) \qquad \langle A, [L]\rangle = \mathrm{tr}(AL), \quad A \in \mathcal{A}_T, [L] \in Q_T.$$

The Banach space Q_T is called a predual of $\mathcal{A}_T$. For x and y in $\mathcal{H}$, we can write $x \otimes y$ for the rank one operator in $\mathcal{C}_1(\mathcal{H})$ defined by

$$(2) \qquad (x \otimes y)(u) = (u, y)x, \qquad \forall u \in \mathcal{H}.$$

The theory of dual algebras is applied to the study of invariant subspaces, dilation theory, and reflexivity. The classes $\mathbb{A}_{m,n}$ (to be defined in section 2) were defined by H. Bercovici, C. Foias and C. Pearcy in [3]. Also these classes are closely related to the study of the theory of dual algebras. Especially, B. Chevreau and C. Pearcy [7] established

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property $E^r_{\theta,\gamma}$ (to be defined in section 2), and researched a relationship with the class $\mathbb{A}_{1,\aleph_0}$.

In theorem 1 of [9], B. Prunaru got a sufficient condition for the membership in the class $\mathbb{A}_{1,\aleph_0}(\rho)$. In theorem 3.2.2 of [8], Hae Gyu Kim got a sufficient condition for the membership in the class $\mathbb{A}^2_{1,1}(\rho)$. In a sequel to this study(theorem 3.2.2 of [8]), in this paper we obtain a sufficient condition for the membership in the class $\mathbb{A}^2_{1,\aleph_0}(\rho)$ by using techniques in [6] and [9].

2. Preliminaries

Before giving the main result, we recall some ideas introduced in [3], [6], [7], [8] and [10]. Recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator. If T_2 is absolutely continuous or acts on the space (0), T will be called an ***absolutely continuous contraction*** . We denote by $\mathcal{A} = \mathcal{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the Sz.-Nagy-Foias functional calculus $\Phi_T : \mathcal{H}^\infty \to \mathcal{A}_T$ is an isometry (see [4, Theorem 4.1]).

If $\mathcal{A}$ is a dual algebra and m, n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$, then $\mathcal{A}$ is said to have property $(\mathbb{A}_{m,n})$ if each $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \leq i < m, 0 \leq j < n, \tag{3}$$

where $\{[L_{ij}]\}_{\substack{0\leq i<m\\ 0\leq j<n}}$ is an arbitrary $m\times n$ array from $Q_{\mathcal{A}}$, has a solution $\{x_i\}_{0\leq i<m}$, $\{y_j\}_{0\leq j<n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. If $\rho \geq 1$ then $\mathcal{A}$ has property $(\mathbb{A}_{m,n}(\rho))$ if for every $s > \rho$ and every $m \times n$ array $\{[L_{ij}]\}_{\substack{0\leq i<m\\ 0\leq j<n}}$ from $Q_{\mathcal{A}}$ such that the rows and columns of the matrix $([L_{ij}])$ are summable, there exist sequences $\{x_i\}_{0\leq i<m}$ and $\{y_j\}_{0\leq j<n}$ from $\mathcal{H}$ that satisfy (3) and also satisfy the following conditions:

$$\|x_i\|^2 \leq s \sum_{0\leq j<n} \|[L_{ij}]\|, 0 \leq i < m, \tag{4}$$

and

$$(5) \qquad \|y_j\|^2 \le s \sum_{0\le i<m} \|[L_{ij}]\|, 0 \le j < n.$$

For brief notation, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$. Furthermore, if m and n are cardinal numbers such that $1 \le m, n \le \aleph_0$, we denote by $\mathbb{A}_{m,n} = \mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$, and similarly for $\mathbb{A}_{m,n}(\rho)$.

Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let $\theta \in [0,1)$. Then $\mathcal{E}^r_\theta(\mathcal{A})$ denotes the set of all $[L]$ in $Q_\mathcal{A}$ such that there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ of vectors from $\mathcal{H}$ satisfying

(i) $\limsup_{i\to\infty} \|[x_i \otimes y_i] - [L]\| \le \theta$,
(ii) $\|x_i\| \le 1, \|y_i\| \le 1, 1 \le i < \infty$,
(iii) $\|[x_i \otimes z]\| \to 0, \forall z \in \mathcal{H}$ and
(iv) $\{y_i\}$ converges weakly to zero.

If $0 \le \theta < \gamma < +\infty$, then a dual algebra $\mathcal{A}$ is said have property $E^r_{\theta,\gamma}$ if the closed absolutely convex hull of the set $\mathcal{E}^r_\theta(\mathcal{A})$ contains the closed ball $B_{0,\gamma}$ of radius γ centered at the origin in $Q_\mathcal{A}$:

$$(6) \qquad \overline{aco}(\mathcal{E}^r_\theta(\mathcal{A})) \supset \{[L] \in Q_\mathcal{A} : \|[L]\| \le \gamma\} = B_{0,\gamma}.$$

We shall employ the notation $C_{\cdot 0} = C_{\cdot 0}(\mathcal{H})$ for the class of all (completely nonunitary) contraction T in $\mathcal{L}(\mathcal{H})$ such that the sequence $\{T^{*n}\}$ converges to zero in the strong operator topology and is denoted by, as usual, $C_{0\cdot} = (C_{\cdot 0})^*$, and $C_{00} = C_{0\cdot} \cap C_{\cdot 0}$.

Recall that every contraction T in $\mathcal{L}(\mathcal{H})$ has a minimal coisometric extension $B = B_T$ that is unique up to unitary equivalence. Given such T and B, one knows that there exists a canonical decomposition of the isometry B^* as

$$(7) \qquad B^* = S \oplus R^*,$$

corresponding to a decomposition of the space

$$(8) \qquad \mathcal{K} = \mathcal{S} \oplus \mathcal{R},$$

where, if $\mathcal{S} \ne (0), S$ is a unilateral shift operator of some multiplicity in $\mathcal{L}(\mathcal{S})$, and, if $\mathcal{R} \ne (0)$, R is a unitary operator in $\mathcal{L}(\mathcal{R})$. Of course, either $\mathcal{S}$ or $\mathcal{R}$ may be (0) ([7]).

LEMMA 2.1. ([7]) *Suppose* $x, y \in \mathcal{H}, z \in \mathcal{K}$ *and* $T(\in \mathbb{A}(\mathcal{H}))$ *has minimal coisometric extension* B *in* $\mathcal{L}(\mathcal{K})$. *Then*

(9) $B \in \mathbb{A}(\mathcal{K}), \Phi_T \circ \Phi_B^{-1}$ *is an isometry and weak* homeomorphism from* $\mathcal{A}_B$ *onto* $\mathcal{A}_T$, *and* $j = \varphi_B^{-1} \circ \varphi_T$ *is a linear isometry of* Q_T *onto* Q_B,

(10) $j([x \otimes y]_T) = [x \otimes y]_B$,

(11) $[x \otimes z]_B = [x \otimes Pz]_B$,

where P *is the projection of* $\mathcal{K}$ *onto the subspace* $\mathcal{H}$.

DEFINITION 2.2. ([8]) Let m, n and l be any cardinal numbers such that $1 \leq m, n, l \leq \aleph_0$. We denote by $\mathbb{A}^l_{m,n}(\mathcal{H})$ the class of all sets $\{T_k\}_{k=1}^l$ such that T_k belongs to $\mathbb{A}(\mathcal{H})$ for all $k = 1, \cdots, l$ and that every $m \times n \times l$ system of simultaneous equations of the form

$$[x_i \otimes y_j^{(k)}]_{T_k} = [L_{ij}^{(k)}]_{T_k}, \tag{12}$$

where $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from Q_{T_k} for each $k = 1, \cdots, l$, has a solution consisting of a pair of sequences $\{x_i\}_{0 \leq i < m}, \{y_j^{(k)}\}_{\substack{0 \leq j < n \\ 1 \leq k \leq l}}$ of vectors from $\mathcal{H}$. Furthermore, if for every doubly indexed family $\{[L_{ij}^{(k)}]_{T_k}\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ of elements of Q_{T_k} for each $k = 1, \cdots, l$, such that the rows and columns of the matrix $(\|[L_{ij}^{(k)}]_{T_k}\|)$ are summable and r is a fixed real number satisfying $r \geq 1$, then we denoted by $\mathbb{A}^l_{m,n}(r)$ the class of all sets $\{T_k\}_{k=1}^l$ such that T_k belongs to $\mathbb{A}(\mathcal{H})$ for all $k = 1, \cdots, l$ and that every $m \times n \times l$ system of simultaneous equations of the form (12) has a solution consisting of a pair of sequences $\{x_i\}_{0 \leq i < m}, \{y_j^{(k)}\}_{\substack{0 \leq j < n \\ 1 \leq k \leq l}}$ of vectors from $\mathcal{H}$ and also satisfy the following conditions, for every $s > r$,

$$\|x_i\|^2 \leq s \sum_{0 \leq j < n} \|[L_{ij}^{(k)}]_{T_k}\|, 0 \leq i < m, 1 \leq k \leq l \tag{13}$$

and

$$\|y_j^{(k)}\|^2 \leq s \sum_{0 \leq i < m} \|[L_{ij}^{(k)}]_{T_k}\|, 0 \leq j < n, 1 \leq k \leq l. \tag{14}$$

3. Main Results

CONVENTION. *In this section we assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ are either simultaneously* (0) *or simultaneously not* (0) *and so are $\mathcal{S}_1$ and $\mathcal{S}_2$.*

LEMMA 3.1. *Suppose $T_k(\in \mathbb{A}(\mathcal{H}))$ with minimal coisometric extension B_k $(\in \mathcal{L}(\mathcal{S}_k \oplus \mathcal{R}_k))$, $B_k \in C_{\cdot 0}(\mathcal{K})$, and for some $0 < \theta_k < \gamma_k \leq 1$, $\mathcal{A}_{T_k}$ has property $E^r_{\theta_k,\gamma_k}$, for each $k = 1,2$. Suppose also that, for each $k = 1,2, 0 < \rho < 1, [L_k] \in Q_{B_k}$, $a \in \mathcal{H}, w_k \in \mathcal{S}_k, b_k \in \mathcal{R}_k$, and $\delta_k > 0$ are given such that*

$$\|[L_k]_{B_k} - [a \otimes (w_k + b_k)]_{B_k}\| < \delta_k. \tag{15}$$

Then there exist $\hat{a} \in \mathcal{H}, \hat{w}_k \in \mathcal{S}_k, \hat{b}_k \in \mathcal{R}_k, k = 1,2$, such that

$$\|[L_k]_{B_k} - [\hat{a} \otimes (\hat{w}_k + \hat{b}_k)]_{B_k}\| < (\frac{\theta}{\gamma})\delta, \tag{16}$$

and

$$\begin{aligned} &\|\hat{a} - a\| < 6(\frac{\delta}{\gamma})^{\frac{1}{2}}, \qquad \|\hat{w}_k - w_k\| < (\frac{\delta_k}{\gamma_k})^{\frac{1}{2}} \\ &\|\hat{b}_k\| < \frac{1}{\rho}\{\|b_k\| + (\frac{\delta_k}{\gamma_k})^{\frac{1}{2}}\}, \qquad k = 1,2, \end{aligned} \tag{17}$$

where

$$\theta = \max_{k=1,2}\{\theta_k\}, \delta = \max_{k=1,2}\{\delta_k\} \quad and \quad \gamma = \min_{k=1,2}\{\gamma_k\}.$$

Proof. By theorem 3.2.1 of [8], it is clear.

THEOREM 3.2. *For $k = 1,2$, suppose $T_k(\in \mathbb{A}(\mathcal{H}))$ with minimal coisometric extension $B_k(\in \mathcal{L}(\mathcal{S}_k \oplus \mathcal{R}_k)), B_k \in C_{\cdot 0}(\mathcal{K})$, and for some $0 \leq \theta_k < \gamma_k \leq 1, \max_{k=1,2}\{\theta_k\} < \min_{k=1,2}\{\gamma_k\}, \mathcal{A}_{T_k}$ has property $E^r_{\theta_k,\gamma_k}$. Suppose also that we are given scalar $\beta > 1$ and sequence $\left\{[L_j^{(k)}]_{B_k}\right\}_{j=1}^{\infty} \subset Q_{B_k}$ are given such that*

$$\|[L_j^{(k)}]_{B_k}\| < \mu_j^{(k)}, j \in \mathbb{N}, \quad with \quad \mu_k = \sum_{j=1}^{\infty}(\mu_j^{(k)})^{\frac{1}{2}} < \infty. \tag{18}$$

Moreover if we set

$$\alpha = \frac{1}{\gamma^{\frac{1}{2}} - \theta^{\frac{1}{2}}}, \quad \mu = \max_{k=1,2}\{\mu_k\},$$

and

$$\theta = \max_{k=1,2}\{\theta_k\}, \quad \gamma = \min_{k=1,2}\{\gamma_k\}.$$

Then there exist $x \in \mathcal{H}$, $\left\{w_j^{(k)}\right\}_{j=1}^{\infty} \subset \mathcal{S}_k$ *and* $\left\{b_j^{(k)}\right\}_{j=1}^{\infty} \subset \mathcal{R}_k, k = 1,2$ *such that*

(19) $$[L_j^{(k)}]_{B_k} = [x \otimes (w_j^{(k)} + b_j^{(k)})]_{B_k}, \quad k = 1,2,$$

(20) $$\|x\| < 6\alpha\mu^{\frac{1}{2}},$$

(21) $$\|w_j^{(k)}\| < \alpha(\mu_j^{(k)})^{\frac{1}{2}} \quad for \quad all \quad j \quad and \quad k = 1,2,$$

and

(22) $$\|b_j^{(k)}\| < \alpha\beta(\mu_j^{(k)})^{\frac{1}{2}} \quad for \quad all \quad j \quad and \quad k = 1,2.$$

Proof. Since if $\mathcal{A}_{T_k}$ has property E_{0,γ_k}^r for each k, it also has property E_{θ_k,γ_k}^r for all $0 < \theta_k < \gamma_k$, the right-hand side of (20), (21), and (22) are continuous functions of θ_k and $\mu_j^{(k)}$, it suffices to treat the case $0 < \theta_k < \gamma_k$. Let us set

$$\epsilon_{j,t} = [\max_{k=1,2}\{\mu_j^{(k)}\}](\frac{\theta}{\gamma})^t$$

for all $j \geq 1, t \geq 0$, let $\{s_n\}$ be a sequence of positive numbers strictly decreasing to $\frac{1}{\beta}$ such that $s_1 = 1$, and let $\rho_n = \frac{s_{n+1}}{s_n}, n \geq 1$. Let $B : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection such that $j \leq j'$ and $t \leq t'$ implies $B(j,t) \leq B(j',t')$. Let $w_{j,0}^{(k)} = 0$ in $\mathcal{S}_k$ and $b_{j,0}^{n,k} = 0$ in $\mathcal{R}_k$, for all $j, n \geq 1$, and $k = 1,2$. We shall now construct,by induction, sequences $\{x_n\} \subset$

$\mathcal{H}$, $\{w^{(k)}_{j,t}\}_{j,t\geq 1} \subset \mathcal{S}_k$ and , for $n \geq 1$, finite sequences $\{b^{n,t}_{j,t}\}_{B(j,t)\leq n} \subset \mathcal{R}_k$ such that

$$(23)\quad \|[L^{(k)}_j]_{B_k} - [x_n \otimes (w^{(k)}_{j,t} + b^{n,k}_{j,t})]_{B_k}\| < \epsilon_{j,t}, \quad B(j,t) \leq n, k = 1,2,$$

$$(24)\qquad \|x_n - x_{n-1}\| < 6\left(\frac{\epsilon_{j,t-1}}{\gamma}\right)^{\frac{1}{2}} \quad for \quad n = B(j,t),$$

$$(25)\qquad \|w^{(k)}_{j,t} - w^{(k)}_{j,t-1}\| < \left(\frac{\epsilon_{j,t-1}}{\gamma}\right)^{\frac{1}{2}},$$

for all $j,t \geq 1$, and

$$(26)\qquad \|b^{n,k}_{j,t}\| < \frac{1}{\rho_n}\left\{\|b^{n-1,k}_{j,t-1}\| + \left(\frac{\epsilon_{j,t-1}}{\gamma}\right)^{\frac{1}{2}}\right\},$$

if $B(j,t) \leq n, k = 1,2$.

For $n = 1 = B(1,1)$, we apply Lemma 3.1, with $[L_k] = [L^{(k)}_j], \delta = \mu_1, \rho = \rho_1, a = 0, b_k = 0, w_k = 0, k = 1,2$, to find $x_1 \in \mathcal{H}, w^{(k)}_{1,1} \in \mathcal{S}_k, s^{1,k}_{1,1} \in \mathcal{R}_k$ so that

$$(27)\qquad \|[L^{(k)}_1]_{B_k} - [x_1 \otimes (w^{(k)}_{1,1} + b^{1,k}_{1,1})]_{B_k}\| < \epsilon_{1,1}, \quad k = 1,2,$$

$$(28)\qquad \|x_1\| < 6\left(\frac{\mu_1}{\gamma}\right)^{\frac{1}{2}},$$

$$(29)\qquad \|w^{(k)}_{1,1}\| < \left(\frac{\mu_1}{\gamma}\right)^{\frac{1}{2}},$$

and

$$(30)\qquad \|b^{1,k}_{1,1}\| < \frac{1}{\rho_1}\left(\frac{\mu_1}{\gamma}\right)^{\frac{1}{2}}, k = 1,2,$$

where

$$\mu_1 = \max_{k=1,2}\{\mu_1^{(k)}\}.$$

Suppose now that vectors $\{x_1, \cdots, x_n\}$ in $\mathcal{H}$, $\{w_{j,t}^{(k)}\}_{B(j,t)\leq n}$ in $\mathcal{S}_k$, and $\{b_{j,t}^{n,k}\}_{B(j,t)\leq n}$ in $\mathcal{R}_k, k = 1,2$, have been chosen so that (23) – (26) are satisfied. Let $n+1 = B(p,q)$. Apply Lemma 3.1 with $[L_k] = [L_p^{(k)}], a = x_n, w_k = w_{p,q-1}^{(k)}, b_k = b_{p,q-1}^{n,k}, \delta = \epsilon_{p,q-1}, \rho = \rho_{n+1}$ to find $x_{n+1} \in \mathcal{H}, w_{p,q}^{(k)} \in \mathcal{S}_k$ and $b_{p,q}^{n+1,k} \in \mathcal{R}_k, k = 1,2$, such that

$$\|[L_p^{(k)}]_{B_k} - [x_{n+1} \otimes (w_{p,q}^{(k)} + b_{p,q}^{n+1,k})]_{B_k}\| < \epsilon_{p,q}, k = 1,2, \tag{31}$$

$$\|x_{n+1} - x_n\| < 6\left(\frac{\epsilon_{p,q-1}}{\gamma}\right)^{\frac{1}{2}}, \tag{32}$$

$$\|w_{p,q}^{(k)} - w_{p,q-1}^{(k)}\| < \left(\frac{\epsilon_{p,q-1}}{\gamma}\right)^{\frac{1}{2}}, \tag{33}$$

and

$$\|b_{p,q}^{n+1,k}\| < \frac{1}{\rho_{n+1}}\left\{\|b_{p,q-1}^{n,k}\| + \left(\frac{\epsilon_{p,q-1}}{\gamma}\right)^{\frac{1}{2}}\right\}, k = 1,2. \tag{34}$$

Hence we have

$$\|[L_j^{(k)}]_{B_k} - [x_{n+1} \otimes (w_{j,t}^{(k)} + b_{j,t}^{n+1,k})]_{B_k}\| < \epsilon_{j,t}, k = 1,2,$$

if $B(j,t) \leq n+1$. Therefore (23) – (26) are satisfied for $n+1$.
It also follows from that (24) and (25) that $\{x_n\}$ and $\{w_{j,t}^{(k)}\}_{t=1}^{\infty}$ are Cauchy sequences for all $j \geq 1, k = 1,2$ and from (26) that the sequences $\{b_{j,t}^{(k)}\}_{t=1}^{\infty}$ are bounded for all $j \geq 1, k = 1,2$, where $b_{j,t}^{(k)} = b_{j,t}^{B(j,t),k}$ for all $j,t \geq 1, k = 1,2$. Without loss of generality, we may suppose that $\{b_{j,t}^{(k)}\}_{t=1}^{\infty}$ converges weakly to some $b_j^{(k)} \in \mathcal{R}_k, k = 1,2$. Also let

$$x = \lim_{n\to\infty} x_{n+1} \quad \text{and} \quad w_j^{(k)} = \lim_{t\to\infty} w_{j,t}^{(k)}$$

for each $j \geq 1$. To prove (25), we first know that

$$\lim_t \|[x \otimes (w_j^{(k)} + b_{j,t}^{(k)})]_{B_k} - [x_{n+1} \otimes (w_{j,t}^{(k)} + b_{j,t}^{(k)})]_{B_k}\| = 0$$

because $\|[x \otimes y]\| \leq \|x\|\|y\|, \|x - x_{n+1}\| \to 0$, and the sequence $\{w_{j,t}^{(k)} + b_{j,t}^{(k)}\}_{t=1}^{\infty}, j \in \mathbb{N}, k = 1,2$, are bounded (being weakly convergent to $w_j^{(k)} + b_j^{(k)}$). Thus, using (23), we obtain

$$\lim_t \|[L_j^{(k)}]_{B_k} - [x \otimes (w_j^{(k)} + b_{j,t}^{(k)})]_{B_k}\| = 0, j \in \mathbb{N}, k = 1,2, \tag{35}$$

and hence for any $X_k \in \mathcal{A}_{B_k}$, we have

$$\begin{aligned}
&< X_k, [x \otimes (w_j^{(k)} + b_j^{(k)})]_{B_k} > \\
&= (X_k x, w_j^{(k)} + b_j^{(k)}) \\
&= \lim_t (X_k x, w_j^{(k)} + b_{j,t}^{(k)}) \\
&= \lim_t < X_k, [x \otimes (w_j^{(k)} + b_{j,t}^{(k)})]_{B_k} > \\
&=< X_k, [L_j^{(k)}]_{B_k} >, j \in \mathbb{N}, k = 1,2, \text{by} \quad (35).
\end{aligned}$$

And, it follows from (24) that

$$\begin{aligned}
\|x\| \leq 6 \sum_{j=1}^{\infty} \sum_{t=0}^{\infty} \left(\frac{\epsilon_{j,t}}{\gamma}\right)^{\frac{1}{2}} &= 6 \sum_{j=1}^{\infty} \sum_{t=0}^{\infty} \left\{ \max_{k=1,2} \left(\frac{\mu_j^{(k)}}{\gamma}\right)^{\frac{1}{2}} \right\} \left(\frac{\theta}{\gamma}\right)^{\frac{t}{2}} \\
&= \frac{6}{1 - \left(\frac{\theta}{\gamma}\right)^{\frac{1}{2}}} \left\{ \max_{k=1,2} \sum_{j=1}^{\infty} \left(\frac{\mu_j^{(k)}}{\gamma}\right)^{\frac{1}{2}} \right\}.
\end{aligned}$$

Similarly, we obtain

$$\|w_j^{(k)}\| \leq \frac{(\frac{\mu_j^{(k)}}{\gamma})^{\frac{1}{2}}}{1 - \left(\frac{\theta}{\gamma}\right)^{\frac{1}{2}}}$$

for all $j \geq 1$ and $k = 1, 2$. Moreover, it follows from (26) that

$$
\begin{aligned}
s_{n+1}\|b_{j,t}^{(k)}\| &\leq s_{B(j,t-1)+1}\|b_{j,t-1}^{(k)}\| + \left(\frac{\epsilon_{j,t-1}}{\gamma}\right)^{\frac{1}{2}} \\
&\leq s_{B(j,1)+1}\|b_{j,1}^{(k)}\| + \sum_{l=1}^{t-1}\left(\frac{\epsilon_{j,l}}{\gamma}\right)^{\frac{1}{2}} \\
&\leq \sum_{l=0}^{t-1}\left(\frac{\epsilon_{j,l}}{\gamma}\right)^{\frac{1}{2}} = \sum_{l=0}^{t-1}\left(\frac{\mu_j^{(k)}}{\gamma}\right)^{\frac{1}{2}}\left(\frac{\theta}{\gamma}\right)^{\frac{l}{2}} \\
&\leq \left(\frac{\mu_j^{(k)}}{\gamma}\right)^{\frac{1}{2}} \frac{1}{1-\left(\frac{\theta}{\gamma}\right)^{\frac{1}{2}}},
\end{aligned}
$$

and hence that

$$
\|b_{j,t}^{(k)}\| \leq \frac{\beta\left(\mu_j^{(k)}\right)^{\frac{1}{2}}}{1-\left(\frac{\theta}{\gamma}\right)^{\frac{1}{2}}}\left(\frac{1}{\gamma}\right)^{\frac{1}{2}}
$$

for all $j \geq 1, t \geq 0$ and $k = 1, 2$.

THEOREM 3.3. *Under the same conditions of Theorem 3.2, we have*

$$
(36) \qquad [L_j^{(k)}]_{T_k} = [x \otimes P(w_j^{(k)} + b_j^{(k)})]_{T_k}, k = 1, 2,
$$

where P is the projection of $\mathcal{K}$ onto the subspace $\mathcal{H}$.
Moreover, the set $\{T_1, T_2\}$ is in the class $\mathbb{A}^2_{1,\aleph_0}(6\sqrt{2}\alpha^2)$, where $\alpha = \frac{1}{\gamma^{\frac{1}{2}} - \theta^{\frac{1}{2}}}$.

Proof. In order to prove the first part of theorem 3.3, set $v_{j,t}^{(k)} = P(w_{j,t}^{(k)} + b_{j,t}^{(k)})$ for each $k = 1, 2, t \in \mathbb{N}$. Since $\{v_{j,t}^{(k)}\}_{t=1}^{\infty}$ is bounded, we may suppose , without loss of generality, that $\{v_{j,t}^{(k)}\}_{t=1}^{\infty}$ is weakly convergent to $v_j^{(k)}$, for each $j \in \mathbb{N}$ and $k = 1, 2$. It also follows from that (24) and (25) that $\{x_n\}$ and $\{w_{j,t}^{(k)}\}_{t=1}^{\infty}$ are Cauchy sequences for all $j \geq 1, k = 1, 2$ and from (26) that the sequences $\{b_{j,t}^{(k)}\}_{t=1}^{\infty}$ are bounded for all $j \geq 1, k = 1,2$, where $b^{(k)}_{j,t} = b^{B(j,t),k}_{j,t}$ for all $j,t \geq 1, k = 1,2$. Without loss of generality, we may suppose that $\{b^{(k)}_{j,t}\}_{t=1}^{\infty}$ converges weakly to some $b^{(k)}_j \in \mathcal{R}_k, k = 1,2$. Also let $x = \lim_{n\to\infty} x_{n+1}$ and $w^{(k)}_j = \lim_{t\to\infty} w^{(k)}_{j,t}$ for each $j \geq 1$. since $\{v^{(k)}_{j,t}\}_{t=1}^{\infty}$ is bounded, we have

$$\begin{aligned}
&\|[x \otimes v^{(k)}_{j,t}] - [x_{n+1} \otimes v^{(k)}_{j,t}]\| \\
&= \|[(x - x_{n+1}) \otimes v^{(k)}_{j,t}]\| \\
&\leq \|x - x_{n+1}\| \|v^{(k)}_{j,t}\| \to 0 \quad \text{as} \quad t \to \infty,
\end{aligned}$$

where $n = B(j,t)$. Also, from (10) and (11) with $j_k = \varphi^{-1}_{B_k} \circ \varphi_{T_k}, k = 1,2$, we have

$$\begin{aligned}
&\|[L^{(k)}_j]_{T_k} - [x_{n+1} \otimes v^{(k)}_{j,t}]_{T_k}\| \\
&= \|j_k([L^{(k)}_j]_{T_k}) - [x_{n+1} \otimes v^{(k)}_{j,t}]_{B_k}\|, \quad \text{by} \quad (10) \\
&= \|j_k([L^{(k)}_j]_{T_k}) - [x_{n+1} \otimes (w^{(k)}_{j,t} + b^{(k)}_{j,t})]_{B_k}\|, \quad \text{by} \quad (11) \\
&< \epsilon_{j,t} = \max_{k=1,2} \left\{\mu^{(k)}_j\right\} (\frac{\theta}{\gamma})^t \to 0 \quad \text{as} \quad t \to \infty, \quad \text{by} \quad (35),
\end{aligned}$$

where $n = B(j,t)$. So

$$\|[L^{(k)}_j]_{T_k} - [x \otimes v^{(k)}_{j,t}]_{T_k}\| \to 0 \quad \text{as} \quad t \to \infty.$$

We now complete to show that $[L^{(k)}_j]_{T_k} = [x \otimes v^{(k)}_j]_{T_k}$, and thus complete the proof: for h in $H^\infty(\mathbb{T})$,

$$\begin{aligned}
< h(T_k), [L^{(k)}_j]_{T_k} > &= \lim_{t\to\infty} < h(T_k), [x \otimes v^{(k)}_{j,t}]_{T_k} > \\
&= \lim_{t\to\infty} (h(T_k)x, v^{(k)}_{j,t}) \\
&= (h(T_k)x, v^{(k)}_j) \\
&=< h(T_k), [x \otimes v^{(k)}_j]_{T_k} > .
\end{aligned}$$

Therefore, we have

$$[L^{(k)}_j]_{T_k} = [x \otimes v^{(k)}_j]_{T_k}, j \in \mathbb{N}, k = 1,2.$$

To that end, let $\{[L_j^{(k)}]_{T_k}\}_{j=1}^{\infty}$ be a sequence from $Q_{T_k}, k = 1,2$, such that $d^{(k)} = \sum_{j=1}^{\infty} d_j^{(k)} < \infty$,where $d_j^{(k)} = \|[L_j^{(k)}]_{T_k}\|, k = 1,2$. Choose $s > 6\sqrt{2}\alpha^2$ and positive scalar $\beta > 1$ such that $6(1+\beta^2)^{\frac{1}{2}}\alpha^2 < s$, and also choose, for each integer j and $k = 1,2, \mu_j^{(k)} > d_j^{(k)}$ such that $6(1+\beta^2)^{\frac{1}{2}}\alpha^2(\mu_j^{(k)})^{\frac{1}{2}} < sd_j^{(k)}$. It follows that $6(1+\beta^2)^{\frac{1}{2}}\alpha^2\mu_k < sd^{(k)}$, where $\mu_k = \sum_{j=1}^{\infty}(\mu_j^{(k)})^{\frac{1}{2}} < \infty$. By the first part of theorem 3.3, there exist a vector $\tilde{a}$ and a sequences $\{y_j^{(k)}\} = \{P(w_j^{(k)} + b_j^{(k)})\}_{j=1,2}^{\infty,2}$ from $\mathcal{H}$ satisfying

$$\begin{aligned}
&[L_j^{(k)}]_{T_k} = [\tilde{a} \otimes y_j^{(k)}]_{T_k},\\
&\|\tilde{a}\| \le 6\alpha\mu^{\frac{1}{2}},\\
&\|y_j^{(k)}\| \le \left(\|w_j^{(k)}\|^2 + \|b_j^{(k)}\|^2\right)^{\frac{1}{2}}\\
&< \left(\alpha^2\mu_j^{(k)} + \beta^2\alpha^2\mu_j^{(k)}\right)^{\frac{1}{2}} = \alpha\left(1+\beta^2\right)^{\frac{1}{2}}\left(\mu_j^{(k)}\right)^{\frac{1}{2}}\\
&\qquad j \in \mathbb{N}, k = 1,2.
\end{aligned}$$

Setting

$$\hat{a} = \frac{1}{\sqrt{6}}\left(1+\beta^2\right)^{\frac{1}{4}}\tilde{a}$$

$$\hat{y}_j^{(k)} = \frac{\sqrt{6}}{(1+\beta^2)^{\frac{1}{4}}}y_j^{(k)}, \quad k = 1,2.$$

Then we have

$$\begin{aligned}
&[L_j^{(k)}]_{T_k} = [\hat{a} \otimes \hat{y}_j^{(k)}]_{T_k},\\
&\|\hat{a}\| < \frac{1}{\sqrt{6}}\left(1+\beta^2\right)^{\frac{1}{4}}\|\tilde{a}\| < \left(sd^{(k)}\right)^{\frac{1}{2}},\\
&\|\hat{y}_j^{(k)}\| < \frac{\sqrt{6}}{(1+\beta^2)^{\frac{1}{4}}}\|y_j^{(k)}\| < \left(sd_j^{(k)}\right)^{\frac{1}{2}},\\
&\qquad j \in \mathbb{N}, k = 1,2.
\end{aligned}$$

Therefore, the set

$$\{T_1, T_2\} \in \mathbb{A}_{1,\aleph_0}^2\left(6\sqrt{2}\alpha^2\right).$$

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References

1. C. Apostol, H. Bercovici, C. Foias and C. Pearcy, *Invariant subspaces, dilation theory, and structure of the predual of a dual algebra. I*, Funct. Anal. **63** (1985), 369-404.
2. H. Bercovici, B.Chevreau, C. Foias and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra. II*, Math. Z. **187** (1984), 97-103.
3. H. Bercovici, C. Foias and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra. I*, Michigan Math. J. **30** (1983), 335-354.
4. ______, *Dual algebras with applications to invariant subspaces and dilation theory*, C.B.M.S. Regional Conference Series, no. 56, A.M.S., Providence, R. I.
5. A. Brown and C. Pearcy, *Introduction to operator theory. I*, Elements of functional analysis, Springer-Verlag, New York, 1977.
6. B. Chevreau, G. Exner, and C. Pearcy, *On the structure of contraction operators, III*, Michigan Math. J. **36** (1989), 29-62.
7. B. Chevreau, and C. Pearcy, *On the structure of contraction operators, I*, J. Funct. Anal., **76** (1988), 1-29.
8. Hae Gyu Kim, *Ph.D. Thesis*, Kyungpook National University, 1992.
9. B. Prunaru, *On the class* $\mathbb{A}_{1,\aleph_0}$, Proceedings of the American Math. Soc. **112** (1991), 45 - 51.
10. B. Sz-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert spaces*, North Holland Akademiai Kiado, Amsterdam/Budapest, 1970.

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