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ISOMETRIES OF $\mathcal{B}_{2n}^{(T_0)}$

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1. Introduction

The study of self-adjoint operator algebras on Hilbert space is well established, with a long history including some of the strongest mathematicians of the twentieth century. By contrast, non-self-adjoint CSL-algebras, particularly reflexive algebras, are only begins to be studied by W. B. Arveson[1] in 1974. One of the most important classes of such algebras is the sequence $\mathcal{A}_2, \mathcal{A}_4, \cdots, \mathcal{A}_\infty$ of tridiagonal algebras, discovered by F. Gilfeather and D. Larson [2]. In [6], Jo obtained characterizations of isometries of tridiagonal algebras $\mathcal{A}_{2n}$ and $\mathcal{A}_\infty$. One another important class is the sequence $\mathcal{B}_2, \mathcal{B}_4, \cdots, \mathcal{B}_\infty$ which was introduced in [7]. In [8], Jo and Ha considered a generalization $\mathcal{A}_{2n}^{(m)}$ of $\mathcal{A}_{2n}$ and characterized isometries of these algebras. In this paper, we will consider a generalization $\mathcal{B}_{2n}^{(T_0)}$ of $\mathcal{B}_{2n}$ and investigate isometric maps of these algebras.

First we will introduce the terminologies which are used in this paper. Let $\mathcal{H}$ be a complex Hilbert space. If x and y are two vectors in $\mathcal{H}$, then (x, y) means the inner productor of the two vectors x and y. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, then $Alg\mathcal{L}$ is the algebra of all bounded operators acting on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$, containing 0 and I. Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra consisting of all bounded operators acting on $\mathcal{H}$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$. A lattice

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$\mathcal{L}$ is commutative if each pair of projections in $\mathcal{L}$ commutes. If $\mathcal{L}$ is a commutative subspace lattice, then $Alg\mathcal{L}$ is called a CSL-algebra. By an isometry of an operator algebra $\mathcal{A}$ we mean a linear map $\varphi : \mathcal{A} \to \mathcal{A}$ such that $\|\varphi(A)\| = \|A\|$ for every A in $\mathcal{A}$. If $x_1, x_2, \cdots, x_m \in \mathcal{H}$, then $[x_1, x_2, \cdots, x_m]$ means the closed subspace of $\mathcal{H}$ generated by the vectors $x_1, x_2, \cdots, x_m$. Let i and j be two nonzero natural numbers. Then E_{ij} is the matrix whose (i,j)-component is 1 and all other entries are zero. In this paper we will introduce a definition as following: An $n \times n$ matrix $A = (a_{ij})$ has a semi-cyclic chain if there is a finite sequence $a_{i_1,j_1}, a_{i_2,j_1}, a_{i_2,j_2}, a_{i_3,j_2}, a_{i_3,j_3}, \cdots, a_{i_{n-1},j_{n-1}}, a_{i_n,j_{n-1}}, a_{i_n,j_n}$ of entries of A such that $a_{i_1,j_1}a_{i_2,j_1}a_{i_2,j_2}a_{i_3,j_2}a_{i_3,j_3}\cdots a_{i_n,j_{n-1}}a_{i_n,j_n} \neq 0$.

Let $\mathcal{D}$ be the set of all $n \times n$ diagonal matrices and let $\mathcal{S}$ be any linear subspace (not necessarily an algebra) of $n \times n$ matrices with properties that $\mathcal{DS} \subset \mathcal{S}$, $\mathcal{SD} \subset \mathcal{S}$ and $\mathcal{S}$ contains a matrix T_0 which has a semi-cyclic chain ($\mathcal{DS} = \{DS : D \in \mathcal{D}, S \in \mathcal{S}\}$). Then the set of all matrices of the form $\begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$, with $D_1, D_2 \in \mathcal{D}$ and $S \in \mathcal{S}$, forms an algebra. We will denote this algebra by $\mathcal{B}_{2n}^{(T_0)}$. Then $\mathcal{B}_{2n}^{(T_0)}$ is a non-self-adjoint reflexive CSL-algebra and a generalization of the algebras $\mathcal{A}_{2n}$, $\mathcal{B}_{2n}$ and $\mathcal{A}_{2n}^{(m)}$[6,7,8]. In this paper, we will prove the following theorem.

THEOREM. *Let $\varphi : \mathcal{B}_{2n}^{(T_0)} \to \mathcal{B}_{2n}^{(T_0)}$ be an isometry such that $\varphi(I) = I$. Then there is a unitary operator W such that $\varphi(A) = WAW^*$ or $\varphi(A) = WA^tW^*$ for all A in $\mathcal{B}_{2n}^{(T_0)}$, where A^t is the transposed matrix of A.*

2. Examples

EXAMPLE 2.1. Let $\mathcal{H}$ be a $2n$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$. Let U and V be unitary operators such that UAV is in $\mathcal{B}_{2n}^{(T_0)}$ for every $A \in \mathcal{B}_{2n}^{(T_0)}$. Define $\varphi : \mathcal{B}_{2n}^{(T_0)} \to \mathcal{B}_{2n}^{(T_0)}$ by $\varphi(A) = UAV$ for all $A \in \mathcal{B}_{2n}^{(T_0)}$. Then φ is an isometry.

EXAMPLE 2.2. Let $\mathcal{H}$ be as in Example 2.1 and let $U = (u_{ii})$ be a $2n \times 2n$ diagonal matrix such that $|u_{ii}| = 1$ for all $i = 1, 2, \cdots, 2n$.

Define $\varphi : \mathcal{B}_{2n}^{(T_0)} \to \mathcal{B}_{2n}^{(T_0)}$ by $\varphi(A) = U^*AU$ for all $A \in \mathcal{B}_{2n}^{(T_0)}$. Then φ is an isometry such that $\varphi(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots, 2n$. If $E_{ij} \in \mathcal{B}_{2n}^{(T_0)}$, then the (i, j)-component of $\varphi(A)$ is $\overline{u}_{ii}a_{ij}u_{jj}$ for $A = (a_{ij}) \in \mathcal{B}_{2n}^{(T_0)}$, where $\overline{u}_{ii}$ is the complex conjugate of u_{ii}.

EXAMPLE 2.3. Let

$$T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \text{ and } S \in \mathcal{S} \text{ iff } S = \begin{pmatrix} * & 0 & * & 0 & * \\ * & * & 0 & 0 & * \\ 0 & * & * & 0 & 0 \\ 0 & * & * & * & 0 \\ * & 0 & 0 & * & * \end{pmatrix}.$$

Let $W = \begin{pmatrix} U & \mathbf{0} \\ \mathbf{0} & V \end{pmatrix}$, where $U = E_{12} + E_{21} + E_{33} + E_{44} + E_{55}$ and $V = E_{15} + E_{23} + E_{32} + E_{44} + E_{51}$. Define $\varphi : \mathcal{B}_{10}^{(T_0)} \to \mathcal{B}_{10}^{(T_0)}$ by $\varphi(A) = W^*AW$ for all $A \in \mathcal{B}_{10}^{(T_0)}$. Then φ is an isometry such that $\varphi(I) = I$, $\varphi(E_{11}) = E_{22}$, $\varphi(E_{22}) = E_{11}$, $\varphi(E_{33}) = E_{33}$, $\varphi(E_{44}) = E_{44}$, $\varphi(E_{55}) = E_{55}$, $\varphi(E_{66}) = E_{10,10}$, $\varphi(E_{77}) = E_{88}$, $\varphi(E_{88}) = E_{77}$, $\varphi(E_{99}) = E_{99}$, and $\varphi(E_{10,10}) = E_{66}$. Let $E_{ij} \in \mathcal{B}_{10}^{(T_0)}$. If $\varphi(E_{ii}) = E_{pp}$ and $\varphi(E_{jj}) = E_{qq}$, then $\varphi(E_{ij}) = E_{pq}$. In this example, let $\varphi(E_{ii}) = E_{tt}$. If $1 \le i \le 5$, then $1 \le t \le 5$. If $6 \le i \le 10$, then $6 \le t \le 10$.

EXAMPLE 2.4. Let

$$T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \text{ and } S \in \mathcal{S} \text{ iff } S = \begin{pmatrix} * & 0 & * & 0 \\ * & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix}.$$

Let W be the 8×8 matrix whose $(i, 9-i)$-component is 1 for all $i (1 \le i \le 8)$ and all other entries are 0. Define $\varphi : \mathcal{B}_8^{(T_0)} \to \mathcal{B}_8^{(T_0)}$ by $\varphi(A) = W^*A^tW$ for all $A \in \mathcal{B}_8^{(T_0)}$, where A^t is the transposed matrix of A. Then φ is an isometry such that $\varphi(I) = I$ and $\varphi(E_{ii}) = E_{9-i,9-i}$ for all $i (1 \le i \le 8)$. Let $E_{ij} \in \mathcal{B}_8^{(T_0)}$. If $\varphi(E_{ii}) = E_{pp}$ and $\varphi(E_{jj}) = E_{qq}$, then $\varphi(E_{ij}) = E_{qp}$.

3. Isometries of $\mathcal{B}_{2n}^{(T_0)}$

THEOREM 3.1. *Let $T_0 = (t_{ij})$ be an $n \times n$ matrix and let T_0 has a semi-cyclic chain $t_{i_1,j_1}, t_{i_2,j_1}, t_{i_2,j_2}, \cdots, t_{i_n,j_n}$. Then there is an $n \times n$ matrix S_0 whose (p,p)-component is t_{i_p,j_p} for all $p = 1, 2, \cdots, n$ and $(q, q-1)$-component is $t_{i_q,j_{q-1}}$ for all $q = 2, 3, \cdots, n$ and an $n \times n$ unitary matrix W such that $W\mathcal{B}_{2n}^{(T_0)}W^* = \mathcal{B}_{2n}^{(S_0)}$, where $W\mathcal{B}_{2n}^{(T_0)}W^* = \{WAW^* : A \in \mathcal{B}_{2n}^{(T_0)}\}$.*

Proof. Let U be an $n \times n$ matrix whose (k, i_k)-component is 1 for all $k (1 \leq k \leq n)$ and all other entries are 0 and let V be an $n \times n$ matrix whose (j_k, k)-component is 1 for all $k(1 \leq k \leq n)$ and all other entries are 0. Put $S_0 = UT_0V^*$. Then the (p,p)-component of S_0 is t_{i_p,j_p} for all $p = 1, 2, \cdots, n$ and the $(q, q-1)$-component of S_0 is $t_{i_q,j_{q-1}}$ for all $q = 2, 3, \cdots, n$. Let $W = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$, then $W\mathcal{B}_{2n}^{(T_0)}W^* = \mathcal{B}_{2n}^{(S_0)}$.

Throughout this paper we will assume that S_0 is an $n \times n$ matrix whose (i,i)-, and $(j, j-1)$-component of S_0 are nonzero for all $i, j (1 \leq i \leq n, 2 \leq j \leq n)$ and we will investigate isometries of $\mathcal{B}_{2n}^{(S_0)}$. Let $\mathcal{H}$ be a $2n$-dimensional complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$. Let $E_{i,i(1)}, E_{i,i(2)}, \cdots, E_{i,i(p_i)}$ be in $\mathcal{B}_{2n}^{(S_0)}$ $(1 \leq i \leq n, n+1 \leq i(1), i(2), \cdots, i(p_i) \leq 2n)$, and let $E_{1(n+j),n+j}$, $E_{2(n+j),n+j}, \cdots, E_{q_j(n+j),n+j}$ be in $\mathcal{B}_{2n}^{(S_0)}$ $(1 \leq j \leq n, 1 \leq 1(n+j), 2(n+j), \cdots, q_j(n+j) \leq n)$.

THEOREM 3.2. *Let $\mathcal{L}$ be the subspace lattice generated by $\{[e_1], [e_2], \cdots, [e_n], [e_{1(n+1)}, e_{2(n+1)}, \cdots, e_{q_1(n+1)}, e_{n+1}], [e_{1(n+2)}, e_{2(n+2)}, \cdots, e_{q_2(n+2)}, e_{n+2}], \cdots, [e_{1(2n)}, e_{2(2n)}, \cdots, e_{q_n(2n)}, e_{2n}]\}$. Then $\mathcal{B}_{2n}^{(S_0)} = Alg\mathcal{L}$ and hence $\mathcal{B}_{2n}^{(S_0)}$ is a non-self-adjoint reflexive CSL-algebra.*

Let x and y be two vectors in a Hilbert space $\mathcal{H}$. Then $x \otimes y$ is a rank one operator defined by $(x \otimes y)(h) = (h, x)y$ for every $h \in \mathcal{H}$.

LEMMA 3.3. ([10]) *Let $\mathcal{L}$ be a subspace lattice and let x and y be two vectors. Then $x \otimes y$ is in $Alg\mathcal{L}$ if and only if there exists E in $\mathcal{L}$ such that y is in E and x is in $E_-^{\perp}$, where $E_- = \vee\{F : F \in \mathcal{L}$ and $F \not\geq E\}$ and $E_-^{\perp} = (E_-)^{\perp}$.*

LEMMA 3.4. ([12]) *Let $\mathcal{L}$ be a subspace lattice and let $\varphi : Alg\mathcal{L} \to Alg\mathcal{L}$ be a surjective isometry. If $\varphi(I) = A$ and $x \otimes x \in Alg\mathcal{L}$, then $\|Ax\| = \|x\|$, where I repersents the identity operator.*

THEOREM 3.5. *Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be a surjective isometry. Then $\varphi(I)$ is a diagonal unitary operator.*

Proof. Let $E_{i,i(1)}, E_{i,i(2)}, \cdots, E_{i,i(p_i)}$ be in $\mathcal{B}_{2n}^{(S_0)}$ $(i = 1, 2, \cdots n)$ and let $\varphi(I) = (b_{ij})$. Since $e_i \otimes e_i \in Alg\mathcal{L}$, $\|\varphi(I)e_i\| = \|e_i\| = 1$ by Lemma 3.3. Since $\varphi(I)e_i = b_{ii}e_i$, $|b_{ii}| = 1$ for all $i = 1, 2, \cdots, n$. Since $\|\varphi(I)\| = \|I\| = 1$, $\|\varphi(I)\| = \|\varphi(I)^*\| = 1$ and $\|\varphi(I)^* e_i\| = |b_{ii}|^2 + |b_{i,i(1)}|^2 + \cdots + |b_{i,i(p_i)}|^2 \leq 1$ for all $i = 1, 2, \cdots, n$. Hence $b_{i,i(1)} = b_{i,i(2)} = \cdots = b_{i,i(p_i)} = 0$ for all $i = 1, 2, \cdots, n$. Also $\|\varphi(I)e_{n+i}\| = |b_{n+i,n+i}| = 1$ for all $i = 1, 2, \cdots, n$. Thus $\varphi(I)$ is a diagonal unitary operator.

Let $\mathcal{D} = \{A : A$ is a diagonal operator in $\mathcal{B}_{2n}^{(S_0)}\}$. Then $\mathcal{D}$ is a masa, maximal abelian self-adjoint subalgebra, containing $\mathcal{L}$ which is introduced in Theorem 3.2 and $\mathcal{D} = (\mathcal{B}_{2n}^{(S_0)}) \cap (\mathcal{B}_{2n}^{(S_0)})^*$, where $(\mathcal{B}_{2n}^{(S_0)})^* = \{A^* : A$ is in $\mathcal{B}_{2n}^{(S_0)}\}$.

LEMMA 3.6. ([9]) *A linear map φ of one C^*-algebra into another which carries the identity into the identity and is isometric on normal operators preserves adjoints, i.e., $\varphi(A^*) = (\varphi(A))^*$.*

DEFINITION 3.7. ([9]) *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^*-algebras. A Jordan isomorphism or C^*-isomorphism $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ is a bijective linear map such that if $A = A^*$ in $\mathcal{A}_1$, then $\varphi(A) = (\varphi(A))^*$ and $\varphi(A^n) = (\varphi(A))^n$.*

LEMMA 3.8. ([9]) a) *A linear bijection φ of one C^*-algebra $\mathcal{A}_1$ onto another $\mathcal{A}_2$ which is isometric is a C^*-isomorphism followed by left multiplication by a fixed unitary operator, viz, $\varphi(I)$.*

b) A C^*-isomorphism φ of a C^*-algebra $\mathcal{A}_1$ onto a C^*-algebra $\mathcal{A}_2$ is isometric and preserves commutativity.

Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be a surjective isometry and let $\varphi(I) = U$. Define $\tilde{\varphi} : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ by $\tilde{\varphi}(A) = U^*\varphi(A)$ for every A in $\mathcal{B}_{2n}^{(S_0)}$. Then $\tilde{\varphi}$ is an isometry such that $\tilde{\varphi}(I) = I$. Since $\mathcal{D}$ is a C^*-algebra, $\tilde{\varphi}(I) = I$, and $\tilde{\varphi}$ is an isometry, $\tilde{\varphi}|\mathcal{D}$ preserves adjoints by Lemma 3.6. From this fact, we can prove the following lemma.

LEMMA 3.9. *$\tilde{\varphi}(\mathcal{D}) = \mathcal{D}$, where $\tilde{\varphi}$ is defined above.*

Since $\tilde{\varphi} : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}(S_0)$ is a surjective isometry, just like φ, and since if $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ is a surjective isometry such that $\varphi(I) = I$, then $\tilde{\varphi} = \varphi$, we now work exclusively with $\tilde{\varphi}$ and drop the symbol " $\sim$ ". Equivalently we assume that $\varphi(I) = I$. Then we can get the following corollary.

COROLLARY 3.10. *If $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ is a surjective isometry such that $\varphi(I) = I$, then $\varphi(\mathcal{D}) = \mathcal{D}$.*

Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be a surjective isometry such that $\varphi(I) = I$. Then since $\varphi|\mathcal{D}$ and $\varphi^{-1}|\mathcal{D}$ are Jordan isomorphisms, we can prove the following lemma.

LEMMA 3.11. *Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be a surjective isometry such that $\varphi(I) = I$. Then E is a projection in $\mathcal{D}$ if and only if $\varphi(E)$ is a projection in $\mathcal{D}$.*

LEMMA 3.12. ([9]) *If φ is a Jordan isomorphism from a C^*-algebra $\mathcal{A}_1$ onto a C^*-algebra $\mathcal{A}_2$, then $\varphi(BAB) = \varphi(B)\varphi(A)\varphi(B)$ for all A, B in $\mathcal{A}_1$.*

Let E and F be orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then a partial order relation $\leq$ is described as follows; $E \leq F$ if and only if $EF = FE = E$. From this definition and the minimal property of projection, we can prove the following theorem.

THEOREM 3.13. *Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be an isometry such that $\varphi(I) = I$. Then $\varphi(E_{ii})$ is rank one for each $i = 1, 2, \cdots, 2n$.*

If we summarize Lemma 3.11 and Theorem 3.13, we can get the following theorem.

THEOREM 3.14. *Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be an isometry such that $\varphi(I) = I$. Then $\varphi(E_{ii}) = E_{jj}$ for all $i = 1, 2, \cdots, 2n$.*

LEMMA 3.15. ([12]) *Let R be an operator and suppose that there is a non-negative number M and a positive number N such that, for all complex number α with $|\alpha| \geq N$, we have $\|R + \alpha I\| \leq M^2 + |\alpha|^2$. Then $R = 0$.*

LEMMA 3.16. ([12]) *Let* $\varphi : \mathcal{B}_{2n}^{(S_0)} = Alg\mathcal{L} \to \mathcal{B}_{2n}^{(S_0)} = Alg\mathcal{L}$ *be an isometry such that* $\varphi(I) = I$. *Let* E *be a projection in* $\mathcal{D}$ *and let* T *be in* $\mathcal{B}_{2n}^{(S_0)}$ *with* $T = ETE^{\perp}$. *Then we have* $\varphi(T) = \varphi(E)\varphi(T)\varphi(E)^{\perp} + \varphi(E)^{\perp}\varphi(T)\varphi(E)$.

Note that $E_{ii} = [e_i]$ for all $i = 1, 2, \cdots, 2n$. From Lemma 3.16, we get the following theorem.

THEOREM 3.17. *Let* $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ *be an isometry such that* $\varphi(I) = I$. *Let* $E_{i,i(1)}, E_{i,i(2)}, \cdots, E_{i,i(p_i)}$ *be in* $\mathcal{B}_{2n}^{(S_0)}$ $(1 \leq i \leq n, n+1 \leq i(1), i(2), \cdots, i(p_i) \leq 2n)$. *Let* $\varphi(E_{ii}) = E_{tt}$ *and let* $\varphi(E_{i(k),i(k)}) = E_{t_k,t_k}$ *for all* $k = 1, 2, \cdots, p_i$. *If* $1 \leq t \leq n$, *then* $n+1 \leq t_k \leq 2n$ *and there exists a complex number* α_{t,t_k} *such that* $|\alpha_{t,t_k}| = 1$ *and* $\varphi(E_{i,i(k)}) = \alpha_{t,t_k} E_{t,t_k}$. *If* $n+1 \leq t \leq 2n$, *then* $1 \leq t_k \leq n$ *and there exists a complex number* $\alpha_{t_k,t}$ *such that* $|\alpha_{t_k,t}| = 1$ *and* $\varphi(E_{i,i(k)}) = \alpha_{t_k,t} E_{t_k,t}$.

Proof. Suppose that $1 \leq t \leq n$. Since for each pair i, j,

$$E_{ij} = E_{jj}^{\perp} E_{ij} E_{jj} = E_{ii} E_{ij} E_{ii}^{\perp},$$

we have

$$\varphi(E_{i,i(k)}) = E_{t_k,t_k}^{\perp} \varphi(E_{i,i(k)}) E_{t_k,t_k} + E_{t_k,t_k} \varphi(E_{i,i(k)}) E_{t_k,t_k}^{\perp} \tag{1}$$

and

$$\varphi(E_{i,i(k)}) = E_{tt} \varphi(E_{i,i(k)}) E_{tt}^{\perp} + E_{tt}^{\perp} \varphi(E_{i,i(k)}) E_{tt} \tag{2}$$

Let $\varphi(E_{i,i(k)}) = (\alpha_{i,j}) \in \mathcal{B}_{2n}(S_0)$. If $1 \leq t_k \leq n$, then $\varphi(E_{i,i(k)}) = \begin{pmatrix} \mathbf{0} & S_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ by (1), where $\mathbf{0}$ is an $n \times n$ zero matrix and

$$S_1 = \begin{pmatrix} & & & & \\ \alpha_{t_k,t_k(1)} & \alpha_{t_k,t_k(2)} & \cdot & \cdot & \cdot & \alpha_{t_k,t_k(p_{t_k})} \\ & & & & \end{pmatrix}$$

is an $n \times n$ matrix such that all other entries are zero.
Also $\varphi(E_{i,i(k)}) = \begin{pmatrix} \mathbf{0} & S_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ by (2), where $\mathbf{0}$ is an $n \times n$ zero matrix and

$$S_2 = \begin{pmatrix} \\ \alpha_{t,t(1)} & \alpha_{t,t(2)} & \cdot & \cdot & \cdot & \alpha_{t,t(p_t)} \\ \\ \end{pmatrix}$$

is an $n \times n$ matrix such that all other entries are zero.
Since φ is injective, $t \neq t_k$. Thus $\varphi(E_{i,i(k)}) = \mathbf{0}$. We get a contradiction. Hence $n+1 \leq t_k \leq 2n$. Let $E_{1(n+j),n+j}$, $E_{2(n+j),n+j}$, $\cdots$, $E_{q_j(n+j),n+j}$ be in $\mathcal{B}_{2n}^{(S_0)}$ $(1 \leq j \leq n, 1 \leq 1(n+j), 2(n+j), \cdots, q_j(n+j) \leq n)$. Then $\varphi(E_{i,i(k)}) = \begin{pmatrix} \mathbf{0} & S_3 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ by (1) and (2), where $\mathbf{0}$ is an $n \times n$ zero matrix and

$$S_3 = \begin{pmatrix} & & \\ & \alpha_{1(t_k),t_k} & \\ & \alpha_{2(t_k),t_k} & \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ & \alpha_{q_j(t_k),t_k} & \\ & & \end{pmatrix}$$

is an $n \times n$ matrix such that $t_k = n+j$ and all other entries are zero. Therefore $\varphi(E_{i,i(k)}) = \alpha_{t,t_k} E_{t,t_k}$ for some complex number α_{t,t_k} and since φ is an isometry, $|\alpha_{t,t_k}| = 1$. Similarly, we can prove the second part of this theorem.

THEOREM 3.18. *Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be an isometry such that $\varphi(I) = I$. Let $\varphi(E_{11}) = E_{ll}$ and $\varphi(E_{ii}) = E_{tt} (1 \leq i \leq n)$. If $1 \leq l \leq n$, then $1 \leq t \leq n$. If $n+1 \leq l \leq 2n$, then $n+1 \leq t \leq 2n$.*

Proof. Define a permutation σ on $\{1,2,\cdots,2n\}$ by $\sigma(p) = q$ if $\varphi(E_{pp}) = E_{qq}$. Suppose $1 \le l \le n$. Since $E_{1,n+1}$ is in $\mathcal{B}_{2n}^{(S_0)}$, $\sigma(n+1) \ge n+1$ by Theorem 3.17. Since $E_{2,n+1}$ is in $\mathcal{B}_{2n}^{(S_0)}$, $\sigma(2) \le n$. Since $E_{2,n+2}$ is in $\mathcal{B}_{2n}^{(S_0)}$, $\sigma(n+2) \ge n+1$. Since $E_{3,n+2}$ is in $\mathcal{B}_{2n}^{(S_0)}$, $\sigma(3) \le n$. Continuing this process, we obtain $\sigma(i) \le n$ for all $i = 1,2,\cdots,n$. Similarly we can prove the second part of this theorem.

LEMMA 3.19. ([6]) *Let U be a unitary operator. Then $\|I+U\| = 2$ if and only if 1 is in $sp(U)$, where I represents the identity operator and $sp(U)$ is the set of all spectral values of U.*

LEMMA 3.20 ([6]). *Let*

$$A = \begin{pmatrix} 1 & & & & & & & 1 \\ 1 & 1 & & & & & & \\ & 1 & 1 & & & & & \\ & & 1 & 1 & & & & \\ & & & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & & \\ & & & & & 1 & 1 & \\ & & & & & & 1 & 1 \end{pmatrix},$$

be an $n \times n$ matrix , where all other entries are zero. Then $\|A\| = 2$ and $\|A^t\| = 2$.

We will omit a proof of the following theorem for it can be easily proven.

THEOREM 3.21. *Let U be a $2n \times 2n$ matrix. Suppose that U^*AU is in $\mathcal{B}_{2n}^{(S_0)}$ for all A in $\mathcal{B}_{2n}^{(S_0)}$. If $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ is a map defined by $\varphi(A) = U^*AU$ for all $A \in \mathcal{B}_{2n}^{(S_0)}$ satisfying $\varphi(E_{ii}) = E_{ii}$ for all $i = 1,2,\cdots,2n$, then U is a diagonal unitary operator.*

THEOREM 3.22. *Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be an isometry such that $\varphi(E_{kk}) = E_{kk}$ for all $k = 1,2,\cdots,2n$. Then there exists a unitary operator U such that $\varphi(A) = U^*AU$ for all A in $\mathcal{B}_{2n}^{(S_0)}$*

Proof. Let $A = (a_{kj})$ be in $\mathcal{B}_{2n}^{(S_0)}$ and let $\varphi(E_{kj}) = \alpha_{kj}E_{kj}$ for all E_{kj} in $\mathcal{B}_{2n}^{(S_0)}$, where $|\alpha_{kj}| = 1$. Let $\alpha_{kj} = e^{i\theta_{kj}}$ for all k,j such that

E_{kj} in $\mathcal{B}_{2n}^{(S_0)}$. Then θ_{kj} is known by α_{kj} for all k, j such that E_{kj} in $\mathcal{B}_{2n}^{(S_0)}$. Let $U = (u_{ll})$ be a $2n \times 2n$ unitary diagonal matrix and let $u_{ll} = e^{i\theta_l}(l = 1, 2, \cdots, 2n)$. Consider U^*AU.

If the linear system: $(*) \cdots$

$$\theta_{1(1)} - \theta_1 = \theta_{1,1(1)}, \theta_{1(2)} - \theta_1 = \theta_{1,1(2)} \cdots, \theta_{1(p_1)} - \theta_1 = \theta_{1,1(p_1)}$$
$$\theta_{2(1)} - \theta_2 = \theta_{2,2(1)}, \theta_{2(2)} - \theta_2 = \theta_{2,2(2)} \cdots, \theta_{2(p_2)} - \theta_2 = \theta_{2,2(p_2)}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\theta_{n(1)} - \theta_n = \theta_{n,n(1)}, \theta_{n(2)} - \theta_n = \theta_{n,n(2)} \cdots, \theta_{n(p_n)} - \theta_n = \theta_{n,n(p_n)}$$

has solusions, then $\varphi(A) = U^*AU$ for every A in $\mathcal{B}_{2n}^{(S_0)}$. Let K be the coefficient matrix of the linear system $(*)$ and let $K^1, K^2, \cdots, K^{2n}$ be the column vectors of K. Let $X = (\theta_{1,1(1)}, \theta_{1,1(2)}, \cdots, \theta_{1,1(p_1)}, \theta_{2,2(1)}, \theta_{2,2(2)}, \cdots, \theta_{2,2(p_2)}, \cdots, \theta_{n,n(1)}, \theta_{n,n(2)}, \cdots, \theta_{n,n(p_n)})^t$. Then the linear system $(*)$ has solutions if and only if

$$rank\ K = rank\ (K^1, K^2, \cdots, K^{2n}, X).$$

It is easy to check that $rank\ K = 2n - 1$. If for each $E_{k,n+l} \in \mathcal{B}_{2n}^{(S_0)}$ with $k \leq l$, $\theta_{k,n+l} = \theta_{k,n+k} - \theta_{k+1,n+k} + \theta_{k+1,n+k+1} - \theta_{k+2,n+k+1} + \cdots + \theta_{l-1,n+l-1} - \theta_{l,n+l-1} + \theta_{l,n+l}$ and if for each $E_{q,n+p} \in \mathcal{B}_{2n}^{(S_0)}$ with $q > p$, $\theta_{q,n+p} = \theta_{q,n+q-1} - \theta_{q-1,n+q-1} + \theta_{q-1,n+q-2} - \theta_{q-2,n+q-2} + \cdots + \theta_{p+1,n+p}$, then $rank\ (K^1, K^2, \cdots, K^{2n}, X) = 2n - 1$.

Let $A = \begin{pmatrix} \mathbf{0} & W_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ be a $2n \times 2n$ matrix, where $\mathbf{0}$ is an $n \times n$ zero matrix and

$$W_1 = \begin{array}{r} \\ \\ k- \\ \\ \\ \\ \\ \\ l- \\ \\ \end{array} \begin{pmatrix} & \overset{k}{|} & & & & \overset{l}{|} \\ & & & & & \\ & 1 & & & & 1 \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \\ & & & & \cdot & 1 \\ & & & & 1 & 1 \\ & & & & & \end{pmatrix}$$

be an $n \times n$ matrix whose all other entries are zero. Then A is in $\mathcal{B}_{2n}^{(S_0)}$ and $\varphi(A) = \begin{pmatrix} \mathbf{0} & W_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, where $\mathbf{0}$ is an $n \times n$ zero matrix and $W_2 =$

$$\begin{pmatrix} & & & & & & \\ & \alpha_{k,n+k} & & & & & \alpha_{k,n+l} \\ & \alpha_{k+1,n+k} & \alpha_{k+1,n+k+1} & & & & \\ & & \alpha_{k+2,n+k+1} & \cdot & & & \\ & & & \cdot\ \cdot & & & \\ & & & & \cdot\ \cdot & & \\ & & & & \cdot & \alpha_{l-1,n+l-1} & \\ & & & & & \alpha_{l,n+l-1} & \alpha_{l,n+l} \\ & & & & & & \end{pmatrix}$$

is an $n \times n$ matrix whose all other entries are zero.

Let

$$B = \begin{pmatrix} 1 & & & & & & & 1 \\ 1 & 1 & & & & & & \\ & 1 & 1 & & & & & \\ & & 1 & 1 & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \\ & & & & & & 1 & 1 \\ & & & & & & & 1 \quad 1 \end{pmatrix}$$

be a $(l-k+1) \times (l-k+1)$ matrix, where all other entries are zero. Note that the $(l-k+1) \times (l-k+1)$ matrix B and the $2n \times 2n$ matrix A have the same norm.

Let

$$V = \begin{pmatrix} \alpha_{k,n+k} & & & & & \alpha_{k,n+l} \\ \alpha_{k+1,n+k} & \alpha_{k+1,n+k+1} & & & & \\ & \alpha_{k+2,n+k+1} & \cdot & & & \\ & & \cdot\ \cdot & & & \\ & & & \cdot\ \cdot & & \\ & & & \cdot & \alpha_{l-1,n+l-1} & \\ & & & & \alpha_{l,n+l-1} & \alpha_{l,n+l} \end{pmatrix}$$

be a $(l-k+1)\times(l-k+1)$ matrix such that $\|V\| = \|\varphi(A)\|$. Since φ preserves norm, $\|A\| = \|\varphi(A)\|$. So $\|B\| = \|V\|$. Since $\|B\| = 2$ by Lemma 3.20, $\|V\| = 2$.

Let

$$U = \begin{pmatrix} \overline{\alpha}_{k,n+k} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \overline{\alpha}_{k+1,n+k+1} & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & \overline{\alpha}_{l-1,n+l-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \overline{\alpha}_{l,n+l} \end{pmatrix}$$

be a $(l-k+1)\times(l-k+1)$ matrix. Then U is unitary and $\|VU\| = \|V\|$. Put $VU = I + W_3$, where

$$W_3 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & a_{1,l-k+1} \\ a_{21} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{32} & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_{l-k,l-k-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{l-k+1,l-k} & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} a_{21} &= \alpha_{k+1,n+k}\overline{\alpha}_{k,n+k}, \\ a_{32} &= \alpha_{k+2,n+k+1}\overline{\alpha}_{k+1,n+k+1}, \\ &\cdots\cdots \\ a_{l-k+1,l-k} &= \alpha_{l,n+l-1}\overline{\alpha}_{l-1,n+l-1}, \\ a_{1,l-k+1} &= \alpha_{k,n+l}\overline{\alpha}_{l,n+l}. \end{aligned}$$

Since W_3 is unitary and $\|V\| = \|VU\| = \|I + W_3\| = 2$, 1 is in $sp(W_3)$. Thus there exists a nonzero vector $x = (x_k, x_{k+1}, \cdots, x_l)^t$ such that

$W_3x = x$. Since

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & a_{1,l-k+1} \\ a_{21} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{32} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{l-k,l-k-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{l-k+1,l-k} & 0 \end{pmatrix} \begin{pmatrix} x_k \\ x_{k+1} \\ x_{k+2} \\ \vdots \\ x_{l-1} \\ x_l \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{k,n+l}\overline{\alpha}_{l,n+l}x_l \\ \alpha_{k+1,n+k}\overline{\alpha}_{k,n+k}x_k \\ \alpha_{k+2,n+k+1}\overline{\alpha}_{k+1,n+k+1}x_{k+1} \\ \vdots \\ \alpha_{l,n+l-1}\overline{\alpha}_{l-1,n+l-1}x_{l-1} \end{pmatrix} = \begin{pmatrix} x_k \\ x_{k+1} \\ x_{k+2} \\ \vdots \\ x_{l-1} \\ x_l \end{pmatrix},$$

we have

$$\begin{aligned} \alpha_{k,n+l}\overline{\alpha}_{l,n+l}x_l &= x_k, \\ \alpha_{k+1,n+k}\overline{\alpha}_{k,n+k}x_k &= x_{k+1}, \\ \alpha_{k+2,n+k+1}\overline{\alpha}_{k+1,n+k+1}x_{k+1} &= x_{k+2}, \\ &\cdots\cdots\cdots \\ \alpha_{l,n+l-1}\overline{\alpha}_{l-1,n+l-1}x_{l-1} &= x_l. \end{aligned}$$

If $x_i = 0$ for some i $(k \le i \le l)$, then $x_k = x_{k+1} = \cdots = x_l = 0$. So $x_i \neq 0$ for every $i = k, k+1, \cdots, l$. Then

$$\alpha_{k,n+l}\overline{\alpha}_{l,n+l}\alpha_{l,n+l-1}\overline{\alpha}_{l-1,n+l-1}\cdots\alpha_{k+2,n+k+1}\overline{\alpha}_{k+1,n+k+1}$$

$$\alpha_{k+1,n+k}\overline{\alpha}_{k,n+k}x_kx_{k+1}\cdots x_l = x_kx_{k+1}\cdots x_l.$$

Hence

$$\alpha_{k,n+l}\overline{\alpha}_{l,n+l}\alpha_{l,n+l-1}\overline{\alpha}_{l-1,n+l-1}\cdots\alpha_{k+2,n+k+1}\overline{\alpha}_{k+1,n+k+1}$$

$$\alpha_{k+1,n+k}\overline{\alpha}_{k,n+k} = 1.$$

Equivalently

$$\theta_{k,n+l} = \theta_{k,n+k} - \theta_{k+1,n+k} + \theta_{k+1,n+k+1} - \theta_{k+2,n+k+1} + \cdots + \theta_{l-1,n+l-1} - \theta_{l,n+l-1} + \theta_{l,n+l}.$$

Similarly we can prove that

$$\theta_{q,n+p} = \theta_{q,n+q-1} - \theta_{q-1,n+q-1} + \theta_{q-1,n+q-2} - \theta_{q-2,n+q-2} + \cdots + \theta_{p+2,n+p+1} - \theta_{p+1,n+p+1} + \theta_{p+1,n+p}.$$

Thus rank $(K, X) = 2n - 1$. Hence $\varphi(A) = U^*AU$ for all A in $\mathcal{B}_{2n}^{(S_0)}$.

THEOREM 3.23. *Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be an isometry such that $\varphi(I) = I$ and $\varphi(E_{kk}) = E_{i_k,i_k}$ for $k = 1, 2, \cdots 2n$. If $1 \leq i_1 \leq n$, then there is a unitary operator V such that $V\varphi(E_{kk})V^* = E_{kk}$ for all $k = 1, 2, \cdots, 2n$ and $V\varphi(E_{k,k(l)})V^* = \alpha_{i_k,i_{k(l)}}E_{k,k(l)}$ for $l = 1, 2, \cdots, p_k$, and for some complex number $\alpha_{i_k,i_{k(l)}}$.*

Proof. Let V be a $2n \times 2n$ matrix whose (k, i_k)-component is 1 for all $k = 1, 2, \cdots, 2n$ and all other entries are 0. Then

$$\begin{aligned} V\varphi(E_{kk})V^* &= VE_{i_k,i_k}V^* = VE_{i_k,k} = E_{kk} \text{ and} \\ V\varphi(E_{k,k(l)})V^* &= V\alpha_{i_k,i_{k(l)}}E_{i_k,i_{k(l)}}V^* = \alpha_{i_k,i_{k(l)}}VE_{i_k,i_{k(l)}}V^* \\ &= \alpha_{i_k,i_{k(l)}}VE_{i_k,k(l)} = \alpha_{i_k,i_{k(l)}}E_{k,k(l)} \end{aligned}$$

for some complex number $\alpha_{i_k,i_{k(l)}}$ $(k = 1, 2, \cdots, 2n, l = 1, 2, \cdots, p_k)$.

THEOREM 3.24. *Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be an isometry such that $\varphi(I) = I$ and $\varphi(E_{kk}) = E_{i_k,i_k}$. If $1 \leq i_1 \leq n$, then there is a unitary operator W such that $\varphi(A) = W^*AW$ for all A in $\mathcal{B}_{2n}^{(S_0)}$.*

Proof. By Theorem 3.23, there is a unitary operator V such that $V\varphi(E_{kk})V^* = E_{kk}$ for all $k = 1, 2, \cdots, 2n$ and $V\varphi(A)V^* \in \mathcal{B}_{2n}^{(S_0)}$ for all A in $\mathcal{B}_{2n}^{(S_0)}$. Define $\varphi_1 : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ by $\varphi_1(A) = V\varphi(A)V^*$ for all A in $\mathcal{B}_{2n}^{(S_0)}$. Then φ_1 is an isometry such that $\varphi_1(E_{kk}) = E_{kk}$ for all $k = 1, 2, \cdots, 2n$. By Theorem 3.22, there is a unitary operator U such that $\varphi_1(A) = U^*AU$ for all A in $\mathcal{B}_{2n}^{(S_0)}$. Since $\varphi_1(A) = U^*AU = V\varphi(A)V^*$ for all A in $\mathcal{B}_{2n}^{(S_0)}$, $\varphi(A) = (V^*U^*)A(UV)$ for all A in $\mathcal{B}_{2n}^{(S_0)}$. Put $UV = W$. Then $\varphi(A) = W^*AW$ for all A in $\mathcal{B}_{2n}^{(S_0)}$.

THEOREM 3.25. *Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be an isometry such that $\varphi(I) = I$ and $\varphi(E_{kk}) = E_{i_k,i_k}$ for $k = 1, 2, \cdots, 2n$. If $n + 1 \le i_1 \le 2n$, then there is a unitary operator V such that $V\varphi(A)^tV^* \in \mathcal{B}_{2n}^{(S_0)}$ for all A in $\mathcal{B}_{2n}^{(S_0)}$ and $V\varphi(E_{kk})^tV^* = E_{kk}$ for all $k = 1, 2, \cdots, 2n$.*

Proof. Let V be a $2n \times 2n$ matrix whose (k, i_k)-component is 1 for all $k = 1, 2, \cdots, 2n$ and all other entries are 0. Then

$$V\varphi(E_{kk})^tV^* = VE_{i_k,i_k}V^* = VE_{i_k,k} = E_{kk}$$

for all $k = 1, 2, \cdots, 2n$. If $E_{k,k(l)}$ is in $\mathcal{B}_{2n}^{(S_0)}(1 \le k \le n,\ 1 \le l \le p_k)$, then $\varphi(E_{k,k(l)}) = \alpha_{i_{k(l)},i_k}E_{i_{k(l)},i_k}$. Since

$$\begin{aligned} V\varphi(E_{k,k(l)})^tV^* &= \alpha_{i_{k(l)},i_k}VE_{i_k,i_{k(l)}}V^* \\ &= \alpha_{i_{k(l)},i_k}E_{k,i_{k(l)}}V^* \\ &= \alpha_{i_{k(l)},i_k}E_{k,k(l)} \end{aligned}$$

for all $k = 1, 2, \cdots, n$ and all $l = 1, 2, \cdots, p_k$, we have $V\varphi(A)^tV^* \in \mathcal{B}_{2n}^{(S_0)}$ for all A in $\mathcal{B}_{2n}^{(S_0)}$.

THEOREM 3.26. *Let $\varphi : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ be an isometry such that $\varphi(I) = I$ and $\varphi(E_{kk}) = E_{i_k,i_k}$ for $k = 1, 2, \cdots, 2n$. If $n + 1 \le i_1 \le 2n$, then there is a unitary operator W such that $\varphi(A) = WA^tW^*$ for all A in $\mathcal{B}_{2n}^{(S_0)}$.*

Proof. By Theorem 3.25, there is a unitary operator V such that $V\varphi(E_{kk})^tV^* = E_{kk}$ for all $k = 1, 2, \cdots, 2n$ and $V\varphi(A)^tV^*$ is in $\mathcal{B}_{2n}^{(S_0)}$ for all A in $\mathcal{B}_{2n}^{(S_0)}$. Define $\varphi_1 : \mathcal{B}_{2n}^{(S_0)} \to \mathcal{B}_{2n}^{(S_0)}$ by $\varphi_1(A) = V\varphi(A)^tV^*$ for all A in $\mathcal{B}_{2n}^{(S_0)}$. Then φ_1 is an isometry such that $\varphi_1(E_{kk}) = E_{kk}$ for all $k = 1, 2, \cdots, 2n$. Then there is a unitary operator U such that $\varphi_1(A) = U^*AU$ for all A in $\mathcal{B}_{2n}^{(S_0)}$. Since $\varphi_1(A) = V\varphi(A)^tV^* = U^*AU$ for all A in $\mathcal{B}_{2n}^{(S_0)}$, $\varphi(A) = (UV)^tA^t(V^*U^*)^t$ for all A in $\mathcal{B}_{2n}^{(S_0)}$. Put $(UV)^t = W$. Then $\varphi(A) = WA^tW^*$ for all A in $\mathcal{B}_{2n}^{(S_0)}$.

From Theorems 3.1, 3.24 and 3.26, we have the following theorem.

THEOREM 3.27. *Let $\varphi : \mathcal{B}_{2n}^{(T_0)} \to \mathcal{B}_{2n}^{(T_0)}$ be an isometry such that $\varphi(I) = I$. Then there is a unitary operator W such that $\varphi(A) = WAW^*$ or $\varphi(A) = WA^tW^*$ for all A in $\mathcal{B}_{2n}^{(T_0)}$.*

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