J. Korean Math. Soc. **32** (1995), No. 3, pp. 553–562

CRITICAL RIEMANNIAN METRICS ON COSYMPLECTIC MANIFOLDS

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1. Introduction

In a Recent paper [3], D. Chinea, M. Delon and J. C. Marrero proved that a cosymplectic manifold is formal and constructed an example of compact cosymplectic manifold which is not a global product of a Kaehler manifold with the circle. In this paper we study the compact cosymplectic manifolds with critical Riemannian metrics.

Let M be an m-dimensional compact orientable Riemannian manifold and $\mu(M)$ be the set of C^∞ Riemannian metric G on M satisfying $\int_M dV_G = 1$, where dV_G is the volume element of M measured by G. For an element G in $\mu(M)$, we assume that $f(k)$ is a scalar field on M determined by G as the contraction of a tensor product of the curvature tensor. Then $H_M[G] = \int_M f(k)dV_G$ defines a mapping $H_M : \mu(M) \to R$. In this case, a critical point of H_M is called a critical Riemannian metric with respect to the field $f(k)$ and denoted by G_H.

The following four kinds of critical Riemannian metrics have been studied by M. Berger [1] and Y. Muto [7] :

$$A_M[G] = \int_M K\ dV_G, \qquad B_M[G] = \int_M K^2\ dV_G,$$
$$C_M[G] = \int_M S^2\ dV_G, \qquad D_M[G] = \int_M R^2\ dV_G,$$

Received July 5, 1994.
1991 AMS Subject Classification: 53C20.
Key words: Critical Riemannian metrics, fibred Riemannian space, cosymplectic manifold.
This research was partially supported by TGRC-KOSEF and the Basic Science Research Institute Programs, Ministry of Education, 1994.

where R, S and K are the Riemannian curvature tensor, Ricci curvature tensor and scalar curvature respectively. The equations of the critical Riemannian metrics are written as follows:

$$\begin{aligned} A_{ji} &= C_A G_{ji}, \qquad & B_{ji} &= C_B G_{ji}, \\ C_{ji} &= C_C G_{ji}, \qquad & D_{ji} &= C_D G_{ji}, \end{aligned} \tag{1.1}$$

where C_A, C_B, C_C, C_D are undetermined constants and A_{ji}, B_{ji}, C_{ji} and D_{ji} are given by

$$A_{ji} = -S_{ji} + \frac{1}{2}K\, G_{ji}, \tag{1.2}$$

$$B_{ji} = 2\nabla_j\nabla_i K - 2(\Delta K)G_{ji} - 2K\, S_{ji} + \frac{1}{2}K^2\, G_{ji}, \tag{1.3}$$

$$\begin{aligned} C_{ji} =& \nabla_j\nabla_i K - \nabla_k\nabla^k S_{ji} - \frac{1}{2}(\Delta K)G_{ji} \\ & - 2R_{jkhi}S^{kh} + \frac{1}{2}S_{kh}S^{kh}G_{ji}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} D_{ji} =& 2\nabla_j\nabla_i K - 4\nabla_k\nabla^k S_{ji} + 4S_{jk}S_i{}^k \\ & - 4R_{jkhi}S^{kh} - 2R_{jkhl}R_i{}^{khl} + \frac{1}{2}R_{khlm}R^{khlm}G_{ji}, \end{aligned} \tag{1.5}$$

where ∇ is the Riemannian connection and ΔK is the Laplacian of K with respect to G on M.

2. Cosymplectic Bochner curvature tensor

Let M be an m-dimensional cosymplectic manifold with structure (ϕ, ξ, η, G), that is, a manifold M which admits a 1-form η, a vector fields ξ, a metric tensor G satisfying

$$\begin{aligned} &\eta(\xi) = 1, \qquad \phi^2 X = -X + \eta(X)\xi, \ \phi\xi = 0, \\ &G(\xi, X) = \eta(X), \qquad G(\phi X, \phi Y) = G(X,Y) - \eta(X)\eta(Y) \end{aligned}$$

and $\Phi(X,Y) = G(\phi X, Y)$ and η are both closed for arbitrary vector fields X and Y on M. The relations

$$\nabla_\xi K = 0 \qquad \text{and} \qquad R(X,Y)\xi = 0$$

are easily verified.

The cosymplectic Bochner curvature B is defined by

$$\begin{aligned}
&B(X,Y,Z,U)\\
=&R(X,Y,Z,U) + \{G(X,U) - \eta(X)\eta(U)\}L(Y,Z)\\
&- \{G(Y,U) - \eta(Y)\eta(U)\}L(X,Z) + \{G(Y,Z) - \eta(Y)\eta(Z)\}L(X,U)\\
&- \{G(X,Z) - \eta(X)\eta(Z)\}L(Y,U) + \Phi(X,U)M(Y,Z)\\
&- \Phi(Y,U)M(X,Z) + \Phi(Y,Z)M(X,U) - \Phi(X,Z)M(Y,U)\\
&- 2\{\Phi(Z,U)M(X,Y) + \Phi(X,Y)M(Z,U)\},
\end{aligned}$$

where we have put

$$\begin{aligned}
&L(X,Y) = -\frac{1}{2(m+2)}[S(X,Y) + \alpha\{G(X,Y) - \eta(X)\eta(Y)\}],\\
&M(X,Y) = -L(X,\phi Y) \qquad \text{and}\\
&\alpha = -\frac{K}{4(m+1)}
\end{aligned}$$

We assume that the cosymplectic Bochner curvature B vanishes identically on M, then we get

(2.1)

$$\begin{aligned}
&R(X,Y,Z,U)\\
=&\frac{1}{m+3}[S(Y,Z)\{G(X,U) - \eta(X)\eta(U)\}\\
&-S(X,Z)\{G(Y,U) - \eta(Y)\eta(U)\}\\
&+S(X,U)\{G(Y,Z) - \eta(Y)\eta(Z)\} - S(Y,U)\{G(X,Z) - \eta(X)\eta(Z)\}]\\
&- \frac{K}{(m+1)(m+3)}[\{G(Y,Z) - \eta(Y)\eta(Z)\}\{G(X,U) - \eta(X)\eta(U)\}\\
&- \{G(X,Z) - \eta(X)\eta(Z)\}\{G(Y,U) - \eta(Y)\eta(U)\}]\\
&- \Phi(X,U)M(Y,Z) + \Phi(Y,U)M(X,Z) - \Phi(Y,Z)M(X,U)\\
&+ \Phi(X,Z)M(Y,U) + 2\{\Phi(Z,U)M(X,Y) + \Phi(X,Y)M(Z,U)\}.
\end{aligned}$$

LEMMA 2.1. *If the cosymplectic Bochner curvature B of the cosymplectic manifold M vanishes identically and M is Einstein, then M is locally Euclidean.*

Proof. From (2.1), we easily get

$$S(X,Y) = \Big\{\frac{1}{m}G(X,Y) - \frac{4}{m(m+3)}\eta(X)\eta(Y)\Big\}K.$$

This equation implies $K = 0$ and that $S = 0$. Then we get $R = 0$ by use of (2.1).

If the scalar curvature vanishes on M, then, from (2.1), we have

(2.2)

$$\begin{aligned} &R(X,Y,Z,U)\\ =&\frac{1}{m+3}[S(Y,Z)\{G(X,U) - \eta(X)\eta(U)\}\\ &-S(X,Z)\{G(Y,U) - \eta(Y)\eta(U)\}\\ &+S(X,U)\{G(Y,Z) - \eta(Y)\eta(Z)\} - S(Y,U)\{G(X,Z) - \eta(X)\eta(Z)\}]\\ &-\Phi(X,U)M(Y,Z) + \Phi(Y,U)M(X,Z) - \Phi(Y,Z)M(X,U)\\ &+\Phi(X,Z)M(Y,U) + 2\{\Phi(Z,U)M(X,Y) + \Phi(X,Y)M(Z,U)\}. \end{aligned}$$

Thus we can state

LEMMA 2.2. *If the cosymplectic Bochner curvature B and the scalar curvature vanish on the cosymplectic manifold M, then the curvature tensor on M is of the form (2.2).*

3. Cosymplectic manifold with critical Riemannian metrics

Let M be an m-dimensional cosymplectic manifold. If the Riemannian metric G is a critical Riemannian metric G_B, G_C or G_D, then, by use of (1.1)-(1.5), the undetermined constants C_B, C_C and C_D given by (1.1) are determined as follows :

$$C_B = \frac{1}{2}K^2 - 2\triangle K, \tag{3.1}$$

$$C_C = \frac{1}{2}S_{ji}S^{ji} - \frac{1}{2}\triangle K, \tag{3.2}$$

$$C_D = \frac{1}{2}R_{kjih}R^{kjih}. \tag{3.3}$$

From (1.1), (1.3) and (3.1), we get

$$\nabla_X \nabla_Y K = K\ S(X,Y).$$

Then

$$\int_M K^2\ dV_{G_B} = \int_M \triangle K\ dV_{G_B}.$$

Thus, applying the Green's Theorem, we have

PROPOSITION 3.1. *In a compact cosymplectic manifold M, G is a critical Riemannian metric G_B on M if and only if the scalar curvature vanishes.*

By use of (1.4) and (3.2), we have

PROPOSITION 3.2. *In a cosymplectic manifold M, G is a critical Riemannian metric G_C on M if and only if the Ricci curvature vanishes.*

If the Riemannian metric G on M is the critical Riemannian metric G_D, then we obtain

$$\triangle K = -R_{kjih}R^{kjih}$$

by use of (1.5) and (3.3). Thus we have

PROPOSITION 3.3. *In a compact cosymplectic manifold M, G is a critical Riemannian metric G_D on M if and only if M is locally Euclidean.*

From (2.1), we easily see that the cosymplectic manifold with vanishing cosymplectic Bochner curvature is locally Euclidean if the Ricci curvature vanishes. Moreover, it is well known that if G is a critical Riemannian metric G_A, then G is an Einstein metric. From these facts and Lemma 2.1, Propositions 3.2 and 3.3, we have

PROPOSITION 3.4. *The compact cosymplectic manifold with vanishing cosymplectic Bochner curvature is locally Euclidean if and only if G is a critical Riemannian metrics G_A or G_C or G_D.*

4. Critical Riemannian metrics in the fibred Riemannian space

Let $\{M, B, G, \pi\}$ be a fibred Riemannian space, that is, $\{M, G\}$ is an m-dimensional total space with projectable metric G, B an n-dimensional base space and $\pi : M \to B$ the projection with maximal rank n. The fibre passing through a point $q \in M$ becomes an $p(= m - n)$-dimensional submanifold of M, which is denoted by F_q or generally F. Throughout this section, the ranges of the indices are as follows :

$$\begin{aligned} &h, i, j, k, l = 1, 2, \cdots, m, \\ &a, b, c, d, e = 1, 2, \cdots, n, \\ &\alpha, \beta, \gamma, \delta, \epsilon = n+1, \cdots, n+p = m, \end{aligned}$$

unless stated otherwise.

Let $\{E^\alpha, C^\beta\}$ be dual to the frame $\{E_b, C_\alpha\}$ of M and denoting $\tilde{R}_{kji}{}^h$ components of the curvature in M, we have the structure equation as follows [2, 4, 5, 8] :

$$\tilde{R}_{dcb}{}^a = R_{dcb}{}^a - L_d{}^a{}_\epsilon L_{cb}{}^\epsilon + L_c{}^a{}_\epsilon L_{db}{}^\epsilon + 2L_{dc}{}^\epsilon L_b{}^a{}_\epsilon, \tag{4.1}$$

$$\tilde{R}_{d\gamma b}{}^\alpha = -{}^*\nabla_d h_\gamma{}^\alpha{}_b + {}^{**}\nabla_\gamma L_{db}{}^\alpha + L_d{}^e{}_\gamma L_{eb}{}^\alpha + h_\gamma{}^\epsilon{}_d h_\epsilon{}^\alpha{}_b, \tag{4.2}$$

$$\tilde{R}_{\delta\gamma\beta}{}^\alpha = \bar{R}_{\delta\gamma\beta}{}^\alpha + h_{\delta\beta}{}^e h_\gamma{}^\alpha{}_e - h_{r\beta}{}^e h_\delta{}^\alpha{}_e, \tag{4.3}$$

where $R_{dcb}{}^a$ and $\bar{R}_{\delta\gamma\beta}{}^\alpha$ are the components of the curvature in B and F respectively, $h = (h_{\beta\alpha}{}^b)$ and $L = (L_{cb}{}^\alpha)$ are the components of the second fundamental tensor and normal connection of each fibre F respectively, and we have put

$$\begin{aligned} \tilde{R}_{dcb}{}^a &= G(\tilde{R}(E_d, E_c)E_b, E^a), \\ \tilde{R}_{d\gamma b}{}^\alpha &= G(\tilde{R}(E_d, C_\gamma)E_b, C^\alpha), \\ \tilde{R}_{\delta\gamma\beta}{}^\alpha &= G(\tilde{R}(C_\delta, C_\gamma)C_\beta, C^\alpha), \\ {}^*\nabla_d h_\beta{}^\alpha{}_b &= \partial_d h_\beta{}^\alpha{}_b - \Gamma^e_{db} h_\beta{}^\alpha{}_\epsilon + Q_{d\epsilon}{}^\alpha h_\beta{}^\epsilon{}_b - Q_{d\beta}{}^\epsilon h_\epsilon{}^\alpha{}_b, \\ {}^{**}\nabla_\gamma L_{cb}^\alpha &= \partial_d h_{cb}{}^\alpha - \Gamma^e_{dc} L_{eb}{}^\alpha - \Gamma^e_{ab} L_{ce}{}^\alpha + Q_{d\epsilon}{}^\alpha L_{cb}{}^\epsilon, \\ Q_{c\beta}{}^\alpha &= P_{c\beta}{}^\alpha - h_\beta{}^\alpha{}_c, \end{aligned}$$

${P_{c\beta}}^{\alpha}$ are local function related to $L_{C_\alpha}C^\beta = {P_{d\alpha}}^{\beta}E^d$ and P^e_{dc} is the Christoffel symbol induced by the metric in B.

Denoting by $\tilde{S}_{ji}$ components of the Ricci tensor in M, then we have [2, 4, 5, 8]

(4.4)
$$\tilde{S}_{cb} = \tilde{S}(E_c, E_b) = S_{cb} - 2{L_{c\epsilon}}^{\epsilon}{L_b}^{e}{}_{\epsilon} - {h_\delta}^{\epsilon}{}_c{h_\epsilon}^{\delta}{}_b + \frac{1}{2}({}^*\nabla_c {h_\epsilon}^{\epsilon}{}_b + {}^*\nabla_b {h_\epsilon}^{\epsilon}{}_c),$$
(4.5)
$$\tilde{S}_{\gamma b} = \tilde{S}(C_\gamma, E_b) = {}^{**}\nabla_c {h_\epsilon}^{\epsilon}{}_b + {}^{**}\nabla_\epsilon {h_b}^{\epsilon}{}_\gamma + {}^*\nabla_e {L_b}^{e}{}_\gamma - 2{h_\gamma}^{\epsilon}{}_e {L_b}^{e}{}_\epsilon,$$
(4.6)
$$\tilde{S}_{\gamma\beta} = \tilde{S}(C_\gamma, C_\beta) = \bar{S}_{\gamma\beta} - h_{\gamma\beta}^{\ \ e}{h_\epsilon}^{\epsilon}{}_e + {}^*\nabla_e h_{\gamma\beta}^{\ \ e} - {L_a}^{e}{}_\gamma {L_e}^{a}{}_\beta,$$

where S_{cb} and $\bar{S}_{\gamma\beta}$ are the components of the Ricci tensor in B and F respectively and we have put

$$^{**}\nabla_\delta h_{\gamma\beta}^{\ \ a} = \partial_\delta h_{\gamma\beta}^{\ \ a} - \bar{\Gamma}^\epsilon_{\delta\gamma} h_{\epsilon\beta}^{\ \ a} - \bar{\Gamma}^\epsilon_{\delta\beta} h_{\gamma\epsilon}^{\ \ a} + {L_e}^{a}{}_\delta h_{\gamma\beta}^{\ \ e}$$

and $\bar{\Gamma}^\gamma_{\beta\alpha}$ is the Christoffel symbol induced by the metric in F.

Let K, K_B and K_F be the scalar curvatures of M, B and each fibre F respectively, we have [2, 4, 5, 8]

(4.7) $$K = K_B + K_F - L_{cb\alpha}L^{cb\alpha} - h_{\gamma\beta e}h^{\gamma\beta e} - {h_\gamma}^{\gamma}{}_e {h_\beta}^{\beta}{}_e + 2\, {}^*\nabla_e h^{\ \ \epsilon}_{\epsilon e}.$$

The present author [5] proved that

THEOREM 4.1. *The almost contact metric structure (ϕ, ξ, η, G) on M is cosymplectic, then*
(1) *B is Kaehlerian with a complex structure J,*
(2) *F is cosymplectic with a cosymplectic structure $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$,*
(3) *$L = 0$,*
(4) *each fibre is minimal in M.*

It is well known that [4]

LEMMA 4.2. *If the structure tensor h and L vanish identically on M, then M is locally a Riemannian product space of the base space and a fibre.*

Assume that the cosymplectic Bochner curvature vanishes on the compact cosymplectic manifold M and the metric G on M is a critical

Riemannian metric G_A or G_C or G_D, then, from the Proposition 3. 4, M is locally Euclidean. So that the Theorem 4. 1 and (4.1)-(4.3) imply $R_{dcb}{}^a = 0$, $h = 0$ and $\bar{R}_{\delta\gamma\beta}{}^{\alpha} = 0$. Taking account of the Proposition 3.4 and Lemma 4. 2, we can state

THEOREM 4.3. *Let the fibred Riemannian space M be a compact cosymplectic manifold with vanishing cosymplectic Bochner curvature. If the metric on M is a critical Riemannian metric G_A or G_C or G_D, then M is locally the product of the two locally Euclidean spaces.*

Next, if the cosymplectic Bochner curvature vanishes on M and the scalar curvature vanishes, then the curvature of M is determined as (2.2). By use of the Theorem 4. 1 and (4.1), we get

(4.8)
$$\begin{aligned}(m+3)R_{dcb}{}^a =&(S_{cb} - h_{\beta\alpha c}h^{\beta\alpha}{}_b)\delta_d^a - (S_{db} - h_{\beta\alpha d}h^{\beta\alpha}{}_b)\delta_c^a \\ &+ (S_d{}^a - h_{\beta\alpha d}h^{\beta\alpha a})g_{cb} - (S_c{}^a - h_{\beta\alpha c}h^{\beta\alpha a})g_{db} \\ &- (S_{ce} - h_{\beta\alpha c}h^{\beta\alpha}{}_e)J_b{}^e J_d{}^a + (S_{de} - h_{\beta\alpha e}h^{\beta\alpha}{}_e)J_b{}^e J_c{}^a \\ &- (S_{de} - h_{\beta\alpha d}h^{\beta\alpha}{}_e)J^{ae}J_{cb} + (S_{ce} - h_{\beta\alpha c}h^{\beta\alpha}{}_e)J^{ae}J_{db} \\ &+ 2(S_{de} - h_{\beta\alpha d}h^{\beta\alpha}{}_e)J_c{}^e J_b{}^a + 2(S_{be} - h_{\beta\alpha b}h^{\beta\alpha}{}_e)J^{ae}J_{dc},\end{aligned}$$

where g_{cb} is the metric components of B. From (4.8), we obtain

$$\begin{aligned}2S_{cb} =& -(m+1)h_{\beta\alpha c}h^{\beta\alpha}{}_b - h_{\beta\alpha e}h^{\beta\alpha e}g_{cb} \\ &+ 3(S_{de} - h_{\beta\alpha d}h^{\beta\alpha}{}_e)J_b{}^e J_c{}^d + K_B g_{cb}\end{aligned} \tag{4.9}$$

and that

$$(n+1)K_B = (m+n+4)h_{\beta\alpha b}h^{\beta\alpha b} \tag{4.10}$$

Moreover, by use of (4. 7), we get

$$K_B + K_F = h_{\beta\alpha b}h^{\beta\alpha b}. \tag{4.11}$$

On the other hand, the Theorem 4.1 and (4. 2) imply

$$(n+4)h_{\beta\alpha b}h^{\beta\alpha b} = -(p-1)K_B - nK_F. \tag{4.12}$$

Taking account of (4. 11) and (4. 12), we obtain

$$K_B = \frac{2(n+2)}{1-p+n} h_{\beta\alpha b} h^{\beta\alpha b}, \tag{4.13}$$

$$K_F = \frac{-(m+3)}{1-p+n} h_{\beta\alpha b} h^{\beta\alpha b} \tag{4.14}$$

when $p \neq n+1$.

If we substitute (4.13) into (4.10), then $p(n+3)h_{\beta\alpha b}h^{\beta\alpha b} = 0$ so that $h = 0$ and hence $K_B = 0$ and $K_F = 0$.

In the case of $p = n+1$, (4.12) is reformed to

$$(n+4)h_{\beta\alpha b}h^{\beta\alpha b} = -n(K_B + K_F). \tag{4.15}$$

The equations (4.11) and (4.15) give rise to $2(n+2)h_{\beta\alpha b}h^{\beta\alpha b} = 0$, that is $h = 0$. Then (4.10) and (4.11) imply $K_B = 0$ and $K_F = 0$. Thus we have

LEMMA 4.4. *Let M be the fibred Riemannian space with cosymplectic structure. If the cosymplectic Bochner curvature and the scalar curvature on M vanish identically, then M is locally the product of B and F, and the scalar curvatures of B and F vanish identically.*

Considering the Theorem 4.1, Lemma 4.4 and (4.3), it is reduced to

$$\begin{aligned}
&(m+3)\bar{R}_{\delta\gamma\beta}{}^{\alpha} \\
&= \bar{S}_{\gamma\beta}(\delta_\delta^\alpha - \bar{\eta}_\delta\bar{\xi}^\alpha) - \bar{S}_{\delta\beta}(\delta_\gamma^\alpha - \bar{\eta}_\gamma\bar{\xi}^\alpha + \bar{S}_\delta{}^\alpha(\bar{g}_{\gamma\beta} - \bar{\eta}_\gamma\bar{\eta}_\beta) \\
&\quad - \bar{S}_\gamma{}^\alpha(\bar{g}_{\delta\beta} - \bar{\eta}_\delta\bar{\eta}_\beta) - \bar{S}_\gamma{}^\lambda\bar{\phi}_{\beta\lambda}\bar{\phi}_\delta{}^\alpha + \bar{S}_\delta{}^\lambda\bar{\phi}_{\beta\gamma}\bar{\phi}_\gamma{}^\alpha \\
&\quad - \bar{S}_{\delta\lambda}\bar{\phi}^{\alpha\lambda}\bar{\phi}_{\gamma\beta} + \bar{S}_{\gamma\lambda}\bar{\phi}^{\alpha\lambda}\bar{\phi}_{\delta\beta} + 2\bar{S}_{\delta\lambda}\bar{\phi}_\gamma{}^\lambda\bar{\phi}_\beta{}^\alpha + 2\bar{S}_{\beta\lambda}\bar{\phi}^{\alpha\lambda}\bar{\phi}_{\delta\gamma},
\end{aligned} \tag{4. 16}$$

where $\bar{g}$ is the induced metric on F. From (4. 16), we easily see that $\bar{S}_{\gamma\beta} = 0$ and that $\bar{R}_{\delta\gamma\beta}{}^{\alpha} = 0$.

Furthermore, if we consider (4.9) and Theorem 4.1, then $S_{cb} = 0$ and that $R_{dcb}{}^{a} = 0$ by use of (4.8). Thus taking account of the Theorem 4.3 and Lemma 4.4, we have

THEOREM 4.5. *Let M be the fibred Riemannian space with cosymplectic structure. If the cosymplectic Bochner curvature and the scalar curvature on M vanish, then M is locally product of the two locally Euclidean spaces.*

The following fact can be directly reduced from Proposition 3. 1.

COROLLARY 4.6. *If M is the compact cosymplectic manifold with vanishing cosymplectic Bochner curvature and the metric G on M is the critical Riemannian metric G_B, then M is locally product of the two locally Euclidean spaces.*

Finally, combining Theorem 4.3 and Corollary 4.6, we have

THEOREM 4.7. *If M is the compact cosymplectic manifold with vanishing cosymplectic Bochner curvature and the metric G on M is one of the critical Riemannian metrics G_A, G_B, G_C and G_D, then M is locally product of the two locally Euclidean spaces.*

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