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COLUMN RANKS AND THEIR PRESERVERS OF GENERAL BOOLEAN MATRICES

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1. Introduction

There is much literature on the study of matrices over a finite Boolean algebra. But many results in Boolean matrix theory are stated only for binary Boolean matrices. This is due in part to a semiring isomorphism between the matrices over the Boolean algebra of subsets of a k element set and the k tuples of binary Boolean matrices. This isomorphism allows many questions concerning matrices over an arbitrary finite Boolean algebra to be answered using the binary Boolean case. However there are interesting results about the general (i.e. nonbinary) Boolean matrices that have not been mentioned and they differ somwhat from the binary case.

In many instances, the extension of results to the general case is not immediately obvious and an explicit version of the above mentioned isomorphism was not well known. In [4], Kirkland and Pullman gave a way to derive results in the general Boolean algebra case via the isomorphism from the binary Boolean algebra case by means of a canonical form derived from the isomorphism.

In [2], Beasley and Pullman compared semiring rank and column rank of the matrices over several semirings. The difference between semiring rank and column rank motivated Beasley and Song to investigate of the column rank preservers of matrices over nonnegative integers [3] and over the binary Boolean algebra [5].

In this paper, we will show the extent of the difference between semiring rank and column rank of matrices over a general Boolean

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algebra and we also obtain characterizations of the linear operators that preserve column ranks of general Boolean matrices.

Let $\mathbb{B}$ be the *Boolean algebra* of subsets of a k element set S_k and $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote the singleton subsets of S_k. We write $+$ for union and denote intersection by juxtaposition; 0 denotes the null set and 1 the set S_k. Under these two operations, $\mathbb{B}$ is a commutative, antinegative semiring (that is, only 0 has an additive inverse); all of its elements, except 0 and 1, are zero-divisors. Let $\mathbb{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{B}$. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well.

For each $m \times n$ matrix A over $\mathbb{B}$, the *p-th constituent* [4] of A, A_p, is the $m \times n(0,1)$-matrix whose (s,t)-th entry is 1 if and only if $a_{st} \supseteq \sigma_p$. Via the constituents, A can be written uniquely as $\sum \sigma_p A_p$, which is called *the canonical form* of A.

It follows from the uniqueness of the decomposition, and the fact that the singletons are mutually orthogonal idempotents, that for all $m \times n$ matrices A, all $n \times r$ matrices B and C, and all $\alpha \in \mathbb{B}$,

(a) $(AB)_p = A_p B_p$, (b) $(B + C)_p = B_p + C_p$, (c) $(\alpha A)_p = \alpha_p A_p$ for all $1 \leq p \leq$k.

2. Boolean rank versus Boolean cloumn rank

The *Boolean rank*, $b(A)$, of a nonzero $A \in \mathbb{M}_{m,n}(\mathbb{B})$ is defined as the least index r such that $A = BC$ for some $B \in \mathbb{M}_{m,r}(\mathbb{B})$ and $C \in \mathbb{M}_{r,n}(\mathbb{B})$. The rank of zero matrix is zero; in the case that $\mathbb{B} = \mathbb{B}_1 = \{0,1\}$, we refer to $b(A)$ as the *binary Boolean rank,* and denote it by $b_1(A)$.

For a binary Boolean matrix A, we have $b(A) = b_1(A)$ by definition.

If $\mathbb{V}$ is nonempty subset of $\mathbb{M}_{r,1}(\mathbb{B})$ that is closed under addition and multiplication by scalars, then $\mathbb{V}$ is called a *vector space* over $\mathbb{B}$. The concepts of "subspace" and of "generating sets" are defined so as to coincide with familar definitions when $\mathbb{B}$ is a field. We'll use the notation $< F >$ to denote the subspace generated by the subset F of $\mathbb{V}$. As with fields, a *basis* for a vector space $\mathbb{V}$ is a generating subset of least cardinality. That cardinality is the *dimension*, dim($\mathbb{V}$), of $\mathbb{V}$.

Since $\mathbb{B}_1$ is canonically identified with the subsemiring $\{0,1\}$ of $\mathbb{B}$, a binary Boolean matrix can be considered as a matrix over both $\mathbb{B}$ and $\mathbb{B}_1$.

The *Boolean column rank*, $c(A)$, of $A \in \mathbb{M}_{m,n}(\mathbb{B})$ is the dimension of the space $< A >$ generated by the columns of A. In the binary Boolean algebra, we denote it by $c_1(A)$ for $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$.

(2.1) It is known [2] that for all $m \times n$ matrices A over $\mathbb{B}$, $0 \leq b(A) \leq c(A) \leq$ n.

(2.2) For any $p \times q$ matrix A over $\mathbb{B}$, the Boolean rank of $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ is $b(A)$ and its Boolean column rank is $c(A)$.

(2.3) The Boolean rank of a matrix is the maximum of the binary Boolean ranks of its constituents. ([4])

Let us start with the relationship between $c(A)$ and $c_1(A)$ for a binary Boolean matrix A when considered in $\mathbb{B}$ and $\mathbb{B}_1$.

LEMMA 2.1. *For any binary Boolean matrix A, we have* $c(A) = c_1(A)$.

Proof. . Since $\mathbb{B}_1$ can be considered as a subsemiring of $\mathbb{B}$, we have $c(A) \leq c_1(A)$. Conversely, if $c(A) = r$, then there exists a basis $B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ of the column space of A such that each $\mathbf{x}_i$ is a linear combination of the columns of A over $\mathbb{B}$. Then the p-th constituents $(\mathbf{x}_1)_p, (\mathbf{x}_2)_p, \ldots, (\mathbf{x}_r)_p$ can generate all columns of A_p over $\mathbb{B}_1$. Hence $c_1(A_p) \leq$r. But $A = A_p$ for all p, so $c_1(A) \leq r = c(A)$.¶

Let $\mu\ (\mathbb{B}, m, n)$ be the largest integer γ such that for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$, $b(A) = c(A)$ if $b(A) \leq \gamma$.

(2.4) It follows from definition of μ and (2.2) that if $c(A) > b(A)$ for some $p \times q$ matrix A, then $\mu(\mathbb{B}, m, n) < b(A)$ for all $m \geq p$ and $n \geq q$.

Beasley and Pullman determined the value of μ on $\mathbb{B}_1$ in [2] as follows;

LEMMA 2.2. ([2])

$$\mu(\mathbb{B}_1, m, n) = \begin{cases} 1 & \textit{if } min(m,n) = 1 \\ 3 & \textit{if } m \geq 3 \textit{ and } n = 3 \\ 2 & \textit{otherwise.} \end{cases}$$

Now we determine the value μ *for a nonbinary Boolean algebra* $\mathbb{B}$.

LEMMA 2.3. *If $\mathbb{B}$ is any Boolean algebra and $m \geq 3$ and $n > 3$,*

$$\mu(\mathbb{B}, m, n) \leq 2.$$

Proof. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then $c(A) = c_1(A)$ by Lemma 2.1. Since the four columns of A consitute a basis for the column space of A over $\mathbb{B}_1, c_1(A) = 4$. Thus $c(A) = 4$. But $b(A) \leq \min\{m, n\} = 3$. The result follows from the property (2.4).¶

LEMMA 2.4. *If $c(A) = r$ and $\sum \sigma_p A_p$ is the canonical form of $A \in \mathbb{M}_{m,n}(\mathbb{B})$ then* $\max\{c_1(A_p) \mid 1 \leq p \leq k\} \leq r$.

Proof. . Assume that $c(A)$=r. Then there exists some basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ for the column space of A. Then the pth constituents $(\mathbf{x}_1)_p, \ldots, (\mathbf{x}_r)_p$ can generate all columns of A_p over $\mathbb{B}_1$. Therefore $c_1(A_p) \leq r$ for all p.¶

We remark that the inequality in Lemma 2.4 may be strict for $r > 1$ as shown in Example 2.1 below.

EXAMPLE 2.1. Let

$$A = \begin{bmatrix} \sigma_1 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

be a matrix over the nonbinary Boolean algebra of the k element set S_k, where σ_1 is a singleton subset of S_k. Then $c(A) = 3$ by the proof of Theorem 2.1 below. But $c_1(A_1) = c_1(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}) = 2$ and $c_1(A_p) = c_1(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}) = 2$ for all $p = 2, 3, \ldots,$ k.¶

LEMMA 2.5. *If $\mathbb{B}$ is any Boolean algebra and $m \geq 1$ and $n \geq 1$, then $\mu(\mathbb{B}, m, n) \geq 1$.*

Proof. . If $c(A) = 1$ then $b(A) = 1$ by the property (2.1). If $b(A) = 1$, then A can be factored as $\mathbf{x}\mathbf{a}^t$ where $\mathbf{a}^t = [a_1, \dots, a_n] \in \mathbf{M}_{1,n}(\mathbb{B})$ and $\mathbf{x} \in \mathbf{M}_{m,1}(\mathbb{B})$. Thus the column space of A can be generated by one column vector $\mathbf{x}$. So $c(A) \leq 1 = b(A)$. On the other hand $b(A) \leq c(A)$ by (2.1).¶

THEOREM 2.1. *For a nonbinary Boolean algebra $\mathbb{B}$,*

$$\mu(\mathbb{B}, m, n) = \begin{cases} 2 & \text{if } 2 = n \leq m \\ 1 & \text{otherwise} \end{cases}$$

Proof. By Lemma 2.5, $\mu(\mathbb{B}, m, n) \geq 1$ for all positive integers m and n. Let $\sigma_1 \in \mathbb{B}$ be a singleton subset of S_k. Consider the matrix

$$A = \begin{bmatrix} \sigma_1 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then the column space of A is

$$\mathbb{V} = \left\{ x \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix} + y \begin{bmatrix} \sigma_1 \\ \sigma_1 \end{bmatrix} + z \begin{bmatrix} \sigma_1 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 1 \end{bmatrix} | z, w \in \mathbb{B}, \text{ and } x, y \in < \sigma_1 > \right\}.$$

Let Ω be any subset of $\mathbb{V}$ generating $\mathbb{V}$. Let

$$\mathbf{a} = \begin{bmatrix} \sigma_1 \\ 1 \end{bmatrix}.$$

If $\mathbf{a} \notin \Omega$, then

$$\mathbf{a} = \begin{bmatrix} x + y + w \\ y + w \end{bmatrix} \text{ for some } w \in \mathbb{B}, x, y \in < \sigma_1 > .$$

Now $1 = y + w$ and $y = 0$ or σ_1, so that $w = 1$ or $1 - \sigma_1$(which is the complement of σ_1). But then $\sigma_1 = x + y + w = 1$ or $1 - \sigma_1$, a contradiction, since $\sigma_1 \neq 1$. Hence $\mathbf{a} \in \Omega$.

Let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

If $\mathbf{b} \notin \Omega$, then

$$\mathbf{b} = \begin{bmatrix} x + y + z\sigma_1 \\ y + z \end{bmatrix} \text{ for some } z \in \mathbb{B}, x, y \in < \sigma_1 > .$$

Since $1 = y + z$ and $y = 0$ or σ_1, we have $z = 1$ or $1 - \sigma_1$. But then $1 = x + y + z\sigma_1 = z\sigma_1 (\leq \sigma_1)$ or σ_1, a contradiction, since $\sigma_1 \neq 1$. Hence $\mathbf{b} \in \Omega$.

Let

$$\mathbf{c} = \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix}.$$

Then $\mathbf{c} \notin \Omega$ would imply that $0 = y + z + w$ for some y, z and w, one of which is nonzero, which is impossible. Hence $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \subseteq \Omega$. Therefore $c(A) = 3$. But $b(A) \leq 2$ by definition. Thus μ $(\mathbb{B}, m, n) \leq 1$ when $m \geq 2$ and $n \geq 3$, by property (2.4). Hence $\mu(\mathbb{B}, m, n) = 1$ for $m \geq 2$ and $n \geq 3$ by Lemma 2.5. Evidently $\mu(\mathbb{B}, m, 1) = 1$ for all $m \geq 1$. If $1 = m \leq n$, then the fact that $< x_1, x_2, \ldots, x_n > = < \sum_{i=1}^{n} x_i >$ implies that $\mu(\mathbb{B}, 1, n) = 1$. If $2 = n \leq m$, then $c(A) = 2$ whenever $b(A) = 2$ by property (2.1). Thus $\mu(\mathbb{B}, m, 2) = 2$ for $m \geq 2$ by Lemma 2.5.¶

3. Linear operators that preserve cloumn rank of the nonbinary Boolean matrices

In this section, we obtain the characterizations of the linear operators that preserve Boolean column rank of the nonbinary Boolean matrices.

A linear operator T on $\mathbb{M}_{m,n}(\mathbb{B})$ is said to *preserve Boolean column rank* if $c(T(A)) = c(A)$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$. It *preserves Boolean column rank* r if $c(T(A)) = r$ whenever $c(A)$=r. For the terms Boolean rank preserver and Boolean rank r preserver, they are defined similarly.

If T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$, for each $1 \leq p \leq k$ define its *p-th constituent*, T_p, by $T_p(B) = (T(B))_p$ for every $B \in \mathbb{M}_{m,n}(\mathbb{B}_1)$. By the linearity of T, we have $T(A) = \sum \sigma_p T_p(A_p)$ for any matrix $A \in \mathbb{M}_{m,n}(\mathbb{B})$.

Since $\mathbb{M}_{n,n}(\mathbb{B})$ is a semiring, we can consider the invertible members of its multiplicative monoid. The permutation matrices (obtained by permuting the columns of I_n, the identity matrix) are all invertible. Since 1 is the only invertible member of the multiplicative monoid of $\mathbb{B}$, the permutation matrices are the only invertible members of $\mathbb{M}_{n,n}(\mathbb{B})$.

LEMMA 3.1. *The Boolean column rank of a Boolean matrix is unchanged by pre- or post-multiplication by an invertible matrix.*

Proof. This follows from the fact that an invertible matrix is just a permutation matrix.¶

LEMMA 3.2. *Suppose T is a linear operator on $\mathbb{M}_{m,n}$ $\mathbb{B}$). If T preserves Boolean column rank r, then each constituent T_p preserves Boolean column rank r on $\mathbb{M}_{m,n}$ $(\mathbb{B}_1)$.*

Proof. Assume that $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$ is a binary Boolean matrix with $c_1(A)=$ r. By Lemma 2.1, we have $c(A) = r$ and $c(\sigma_p A) = r$ for each $p = 1, \ldots,$ k. Since T preserves Boolean column rank $r, c(T(\sigma_p A))=$ r. But

$$c(T(\sigma_p A)) = c(\sigma_p T(A)) = c(\sigma_p \sum_i \sigma_i T_i(A_i)) = c(\sigma_p T_p(A))$$

for each p. Therefore $c(\sigma_p T_p(A)) = r$ for each $p = 1, \ldots, k$, and hence $c_1(T_p(A)) = r$.¶

LEMMA 3.3. *Suppose T is a linear operator on the $m \times n$ matrices over $\mathbb{B}$. If each constituent T_p preserves binary Boolean rank r, then T preserves Boolean rank r.*

Proof. Let $b(A) = r$ for $A \in \mathbb{M}_{m,n}(\mathbb{B})$. Then there exists some p such that $b_1(A_p) = r$ and $b_1(A_q) \leq r$ for $1 \leq q \leq k$ by property (2.3). Thus $b_1(T_p(A_p)) = r$ and $b_1(T_q(A_q)) \leq r$ for $1 \leq q \leq$k. Since $b(T(A)) = \max\{b_1(T_p(A_p)) \mid 1 \leq p \leq k\}$ by property (2.3), T preserves Boolean rank r.

Now we need the following definitions of linear operators on the $m \times n$ matrices over $\mathbb{B}$. For any fixed pair of invertible $m \times m$ and $n \times n$ Boolean matrices U and V, the operator $A \to$ UAV is called a *congruence operator*. Let σ^* denote the complement of σ for each σ in $\mathbb{B}$. For $1 \leq p \leq k$, we define the p-th *rotation operator*, $R^{(p)}$, on the $n \times n$ matrices over $\mathbb{B}$ by

$$R^{(p)}(A) = \sigma_p A_p^t + \sigma_p^* A,$$

where A_p^t is the transpose matrix of A_p. We see that $R^{(p)}$ has the effect of transposing A_p while leaving the remaining constituents unchanged. Each rotation operator is linear on the $n \times n$ matrices over $\mathbb{B}$ and their product is the *transposition operator*, $R : A \to A^t$.

EXAMPLE 3.1. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ \sigma_1 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

be a matrix over $\mathbb{B}$. Then $c(A) = 3$ by Example 2.1 and property (2.2). But $R^{(1)}(A) = A^t$, the transpose matrix of A, has Boolean column rank 2. Consider $B = A \oplus 0_{n-3,n-3}$ for $n \geq 3$. By property (2.2), the rotation operator does not preserve Boolean column rank 3 on $\mathbb{M}_{n,n}(\mathbb{B})$.¶

LEMMA 3.4. ([4]) *If T is a linear operator on the $m \times n$ matrices $(m, n \geq 1)$ over a general Boolean algebra $\mathbb{B}$, then the followings are equivalent.*

(1) *T preserves Boolean ranks 1 and 2.*
(2) *T is in the group of operators generated by the congruence (if $m = n$, also the rotation) operators.*

THEOREM 3.1. *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ for $m \geq 3$ and $n \geq 1$. Then the following are equivalent.*

(1) *T preserves Boolean column rank.*
(2) *T preserves Boolean column ranks 1, 2 and 3*
(3) *T is a congruence operator.*

Proof. Obviously (1) implies (2). Assume that T preserves Boolean column ranks 1, 2 and 3. Then each constituent T_p preserves binary Boolean column ranks 1, 2 and 3 by Lemma 3.2. For $A \in \mathbb{M}_{m,n}(\mathbb{B})$, Lemma 2.2 implies that $b_1(A) = c_1(A)$ for $b_1(A) \leq 2$. Thus T_p preserves binary Boolean ranks 1 and 2. Then T preserves Boolean ranks 1 and 2 by Lemma 3.3. So T is in the group of operators generated by the congruence (if $m = n$, also the rotation) operators. But the rotation operator does not preserve Boolean column rank 3 by Example 3.1. Hence T is a congruence operator since T preserves Boolean column rank 3. That is, (2) implies (3). Now, assume that T is a congruence operator of the form $T(A)$=UAV, where U and V are invertible $m \times m$ and $n \times n$ Boolean matrices respectively. Then T preserves Boolean column rank by Lemma 3.1. Hence (3) implies (1). ¶

If $m \leq 2$, then the linear operators that preserve column rank on $\mathbb{M}_{m,n}(\mathbb{B})$ are the same as the Boolean rank-preservers, which were characterized in [4].

Thus we have characterizations of the linear operators that preserve the Boolean column rank of general Boolean matrices.

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