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MORSE INEQUALITY FOR FLAT BUNDLES

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1. Introduction

Let M be a compact smooth manifold of dimension n and let E be a flat (complex) vector bundle over M of rank r. The space of differential forms on M with values in E is denoted by $A^\bullet(E) := \oplus_{k\geq 0} A^k(E)$ and the cohomology spaces of the elliptic complex

$$0 \to A^0(E) \xrightarrow{D} A^1(E) \to \cdots \to A^n(E) \to 0 \tag{1}$$

associated to the canonical flat connection D for E, will be denoted by $H^\bullet(E) := \oplus_{k\geq 0} H^k(E)$. Let $b_{E,k} := \dim_{\mathbb{C}} H^k(E)$. These numbers are generalizations of the ordinary Betti numbers $b_k = \dim_{\mathbb{C}} H^k(M, \mathbb{C})$ and associated to a linear representation of the fundamental group $\pi_1(M)$.

Let

$$b_E(t) := \sum_{k\geq 0} b_{E,k} t^k$$

be the generalized *Poincaré polynomial* associated to the flat bundle E, or the representation of $\pi_1(M)$. Then the Atiyah-Singer index formula [AS] implies that the Euler-Poincaré characteristic of E is equal to r times the Euler-Poincaré characteristic of M;

$$\chi(E) = r\chi_M. \tag{2}$$

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On the other hand, if $f : M \to \mathbb{R}$ is a Morse function and $c_{f,k}$ denotes the number of critical points of f of index k, then we have the *Morse polynomial*

$$c_f(t) := \sum_{k \geq 0} c_{f,k} t^k.$$

Now the classical Morse theory [Bott] says that there exists a polynomial $q_f(t) \in \mathbb{Z}[t]$ with nonnegative coefficients such that

$$c_f(t) - b(t) = (1+t) q_f(t)$$

where $b(t)$ is the ordinary Poincaré polynomial of M. Motivated with this 'inequality' and the identity (2), we were able to deduce the following result. Experts may have this result already but so far we cannot find the published statement.

THEOREM 1.1. *Let f be a Morse function on a compact manifold M of dimension n and let E be a flat vector bundle over M of rank r. Then there exists a polynomial $q(t) \in \mathbb{Z}[t]$ with nonnegative coefficients such that*

$$r c_f(t) - b_E(t) = (1+t) q(t).$$

Note that the above identity is equivalent to the following inequalities

$$c_{f,0} \geq \frac{1}{r} b_{E,0}$$

$$c_{f,1} - c_{f,0} \geq \frac{1}{r}(b_{E,1} - b_{E,0})$$

$$\cdots$$

$$c_{f,n} - c_{f,n-1} + \cdots + (-1)^n c_{f,0} = \frac{1}{r}(b_{E,n} - b_{E,n-1} + \cdots + (-1)^n b_{E,0})$$

where f is an arbitrary Morse function on M and E is an arbitrary flat vector bundle over M of rank r. Note that the left hand side of the above inequality depends only on the function f and the right hand side depends only on the flat bundle E. As a corollary,

COROLLARY 1.2. *For any integer* $k = 0, 1, \ldots, n$,

$$c_{f,k} \geq \frac{1}{r} b_{E,k}.$$

In particular, the number of critical points of a Morse function f *is greater than or equal to* $\frac{1}{r}\sum_{k\geq 0} b_{E,k}$ *for any flat vector bundle* E.

For the proof of Theorem 1.1, we follow the idea of Witten's Morse theory [Wit] proved by Roe [Roe].

Alternatively one may use the gradient flow method of Thom-Smale-Witten-Floer [AB, F]. Namely, if C^k denotes the direct sum of the fibers E_x, where $x \in M$ runs through all critical points of index k, then one can define a boundary operator

$$\partial : C^k \to C^{k+1}$$

using 'parallel translations' along the gradient lines joining the critical points and show its cohomology spaces are equal to $H^k(E)$.

2. Morse inequalities for elliptic complex

Let $E^0, E^1, \ldots, E^l$ be hermitian vector bundles over a compact *Riemannian* manifold M and let

$$(3) \qquad P : 0 \to \mathcal{C}^\infty(E^0) \xrightarrow{P^0} \mathcal{C}^\infty(E^1) \xrightarrow{P^1} \ldots \xrightarrow{P^{l-1}} \mathcal{C}^\infty(E^l) \to 0$$

be an elliptic complex, where $\mathcal{C}^\infty$ denotes the space of smooth sections. The cohomology spaces and their dimensions of this complex will be denoted by $H^k(P)$ and $b_{P,k}$, respectively. We put

$$b_P(t) := \sum_{k\geq 0} b_{P,k} t^k.$$

Using the global inner products

$$\langle\langle \xi_1, \xi_2 \rangle\rangle := \int_M \langle \xi_1, \xi_2 \rangle \, d\,\mathrm{vol}, \qquad \xi_1, \xi_2 \in \mathcal{C}^\infty(E^k),$$

$d\,\mathrm{vol}$ being the canonical density on the Riemannian manifold M, we have the formal adjoint $(P^k)^*$ of P^k and the "Laplacian"

$$\square^k := P^{k-1}(P^{k-1})^* + (P^k)^* P^k : C^\infty(E^k) \to C^\infty(E^k), \quad k = 0, \ldots, l.$$

These Laplacians are positive semi-definite elliptic operators with discrete nonnegative eigenvalues and the exponentials

$$e^{-\square^k} : L^2(E^k) \to L^2(E^k), \quad k = 0, 1, \ldots, l$$

are bounded linear operators with smooth kernel and hence their traces $\mu_{P,k} := \mathrm{Tr}\, e^{-\square^k}$ are well defined.[1] Let

$$\mu_P(t) := \sum_{k \geq 0} \mu_{P,k} t^k.$$

Then the following lemma says that the *Morse polynomial* $\mu_P(t)$ *of an elliptic operator* P *dominates the Euler-Poincaré polynomial* $b_P(t)$ *of* P.

LEMMA 2.1. *There exists a polynomial* $q(t) \in \mathbb{R}[t]$ *with nonnegative coefficients such that*

$$\mu_P(t) - b_P(t) = (1 + t) q(t).$$

Proof. If $\Gamma_\lambda^k \subset C^\infty(E^k)$ denotes the λ-eigenspace of $\square^k$ and $\gamma_\lambda^k := \dim \Gamma_\lambda^k$, for each $\lambda \in \mathbb{R}$, then

$$\mathrm{Tr}\, e^{-\square^k} = \sum_{\lambda \geq 0} e^{-\lambda} \gamma_\lambda^k.$$

Note that $\gamma_0^k = b_{P,k}$ by Hodge theory and

$$0 \to \Gamma_\lambda^0 \xrightarrow{P^0} \Gamma_\lambda^1 \xrightarrow{P^1} \ldots \xrightarrow{P^{l-1}} \Gamma_\lambda^l \to 0$$

[1]Instead of the exponential function, we may take any smooth rapidly decreasing nonnegative function $\phi : \mathbb{R}^+ \to \mathbb{R}$ with $\phi(0) = 1$.

is exact for $\lambda > 0$. Thus if $q_\lambda^{k+1} := \dim P^k(\Gamma_\lambda^k) \geq 0$, then for $\lambda > 0$

$$\gamma_\lambda^k = q_\lambda^k + q_\lambda^{k+1}.$$

Now

$$\begin{aligned}
\mu_P(t) - b_P(t) &= \sum_{k\geq 0} (\mu_{P,k} - b_{P,k}) t^k \\
&= \sum_k \left(\sum_{\lambda>0} e^{-\lambda} \gamma_\lambda^k \right) t^k \\
&= \sum_k \left(\sum_{\lambda>0} e^{-\lambda} (q_\lambda^k + q_\lambda^{k+1}) \right) t^k \\
&= (1+t) \sum_k \left(\sum_{\lambda>0} e^{-\lambda} q_\lambda^{k+1} \right) t^k
\end{aligned}$$

This completes the proof.

3. Asymptotic behavior of smoothing kernel

Let F be a hermitian vector bundle over a compact Riemannian manifold M and let

$$Q : \mathcal{C}^\infty(F) \to \mathcal{C}^\infty(F)$$

be a positive semi-definite (self-adjoint) elliptic operator of order $l > 0$. Then

LEMMA 3.1. *Let U be an open subset of M and $C > 0$. Suppose*

$$\langle \langle Qs, s \rangle \rangle \geq C \|s\|^2$$

for any $s \in \mathcal{C}^\infty(F)$ supported in U. If k is an integer with $0 \leq k \leq C+1$, then

$$\|(1+Q)^k e^{-Q} s\| \leq (1+C)^k e^{-C} \|s\|$$

for any $s \in \mathcal{C}^\infty(F)$ supported in U.

Proof. Let $\mathcal{H}$ be the closure of $\{s \in \mathcal{C}^\infty(F) : \operatorname{supp} s \subset U\}$ in $L^2(F)$. Then Q is a positive formally self-adjoint unbounded operator on $\mathcal{H}$, and by Friedrich's extension theorem Q has a self-adjoint extension $\bar{Q}$ satisfying the same positive condition. Now by the spectral theorem, the operator norm of $(1+\bar{Q})^k e^{-\bar{Q}}$ is bounded by $(1+C)^k e^{-C}$.

Let

$$J_1, J_2 : F \to F$$

be self-adjoint vector bundle homomorphisms, where J_2 is positive semi-definite. Let

$$Q_t := Q + tJ_1 + t^2 J_2$$

for $t \in \mathbb{R}$. We will assume that

$$Q_t : \mathcal{C}^\infty(F) \to \mathcal{C}^\infty(F)$$

is a positive semi-definite elliptic operator so that e^{-Q_t} is defined. Let

$$K_t : M \times M \ni (x,y) \mapsto K_t(x,y) \in \mathrm{Hom}(F_y, F_x)$$

be the smoothing kernel of e^{-Q_t} and hence

$$e^{-Q_t} s(x) = \int_M K_t(x,y) s(y)\, d\,\mathrm{vol}(y), \quad \forall\, s \in \mathcal{C}^\infty(F).$$

Note that by the elliptic estimate, for any nonnegative integer m, there exists a constant $c_m > 0$ such that

$$\|s\|_{m+l} \le c_m(\|Qs\|_m + \|s\|_m), \quad \forall\, s \in \mathcal{C}^\infty(F),$$

where $\|\ \|_m$ denotes the norm of the Sobolev space $L^2_m(F)$. Thus the norm of

$$(1+Q_t)^{-k} : L^2_m(F) \to L^2_{m+kl}(F)$$

is bounded by a polynomial in t.

Suppose J_2 is singular on a subset S of M. For any $\epsilon > 0$, let

$$B_\epsilon := \{x \in M \mid \mathrm{dist}(x,S) \le \epsilon\}.$$

LEMMA 3.2.

$$\lim_{t\to\infty} \sup\{|K_t(x,y)| : x, y \in M - B_\epsilon\} = 0.$$

Proof. Since M is compact, there exists a constant $C > 0$ such that

$$\langle J_2 s(x), s(x)\rangle \geq 2C|s(x)|^2, \qquad x \in M - B_\epsilon,\ s \in \mathcal{C}^\infty(F).$$

Now if t is large, for any $s \in \mathcal{C}^\infty(F)$ supported in the complement of B_ϵ,

$$\langle\langle Q_t(s), s\rangle\rangle = \langle\langle Qs, s\rangle\rangle + t\langle\langle J_1 s, s\rangle\rangle + t^2\langle\langle J_2 s, s\rangle\rangle \geq Ct^2\|s\|^2.$$

By the above lemma, for a given positive integer k, if t is large, we have

$$\|(1+Q_t)^k e^{-Q_t} s\| \leq (1+Ct^2)^k e^{-Ct^2}\|s\|$$

for any $s \in \mathcal{C}^\infty(F)$ supported in the complement of B_ϵ.

Let L^p_k be the closure of $\{s \in \mathcal{C}^\infty(F) : \operatorname{supp} s \subset M - B_\epsilon\}$ in the Sobolev space $L^p_k(F)$ and $L^p = L^p_0$.

Note that the sup norm of K_t is estimated by the operator norm of

$$e^{-Q_t} : L^1 \to L^\infty$$

which is the composition

$$L^1 \xrightarrow{(1+Q_t)^{-k}} L^2 \xrightarrow{(1+Q_t)^{2k}e^{-Q_t}} L^2 \xrightarrow{(1+Q_t)^{-k}} L^2_{kl} \hookrightarrow L^\infty$$

for large k by the Sobolev embedding. Since the norm of

$$(1+Q_t)^{-k} : L^2 \to L^2_{kl} \hookrightarrow L^\infty$$

is bounded by a polynomial in t, the norm of the dual operator

$$(1+Q_t)^{-k} : L^1 \to L^2$$

is also bounded by a polynomial in t. Thus the operator norm of $e^{-Q_t} : L^1 \to L^\infty$ is bounded by $p(t)e^{-Ct^2}$ for some polynomial $p(t)$. Now the result follows from this.

4. Proof of the theorem 1.1

We fix an arbitrary Riemannian metric on M which is flat near the set $S := \operatorname{Crit} f$ of the critical points of f.

Choose any hermitian structure for E and let D^* be the adjoint of the flat connection D in the complex (complex). Then we have the Laplacians

$$\Delta^k := D \circ D^* + D^* \circ D : A^k(E) \to A^k(E)$$

for each $k = 0, 1, \ldots, n$. The space of harmonic sections, i.e., the kernel of Δ^k is, by Hodge theory, isomorphic to $H^k(E)$.

Now we change the given connection D using gauge transformations $e^{tf} : E \to E$ for $t \in \mathbb{R}$ and get

$$D_t := e^{-tf} \circ D \circ e^{tf} = D + t\,\operatorname{ext}(df)$$

and

$$D_t^* = D^* + t\,\operatorname{int}(df),$$

where $\operatorname{ext}(df)$ and $\operatorname{int}(df)$ denote the exterior multiplication and interior multiplication associated to the 1-form df. Our sign convention for the interior multiplication is such that $\operatorname{int}(df)$ is the + adjoint of $\operatorname{ext}(df)$, so that if

$$H := \operatorname{ext}(df) + \operatorname{int}(df)$$

then H^2 is just the multiplication by $|df|^2$.

Now for each $t \in \mathbb{R}$ we have the new Laplacian

$$\Delta_t^k = \Delta^k + tJ + t^2 H^2 : A^k(E) \to A^k(E).$$

where $J := D \circ \operatorname{int}(df) + \operatorname{int}(df) \circ D + \operatorname{ext}(df) \circ D^* + D^* \circ \operatorname{ext}(df)$ is a *self-adjoint endomorphism* of the vector bundle $E^k := (\wedge^k T^* M) \otimes E$. Note that the kernel of Δ_t^k is isomorphic to the kernel of Δ^k, which is again isomorphic to $H^k(E)$.

With

$$\mu_k(t) := \operatorname{Tr}(e^{-\Delta_t^k}), \quad k = 0, 1, \ldots, n,$$

we have, as a corollary of lemma (2.1),

LEMMA 4.1. *For every $t \in \mathbb{R}$,*

$$\mu_0(t) \geq b_{E,0}$$
$$\mu_1(t) - \mu_0(t) \geq b_{E,1} - b_{E,0}$$
$$\dots$$
$$\mu_n(t) - \mu_{n-1}(t) + \cdots + (-1)^n \mu_0(t) = b_{E,n} - b_{E,n-1} + \cdots + (-1)^n b_{E,0}$$

Thus all we have to show is that $\mu_k(t)$ converges to $rc_{f,k}$ as $t \to \infty$.

Take $\epsilon > 0$ small enough so that the metric on M is flat on

$$B_\epsilon = \{x \in M \mid \operatorname{dist}(S, x) \leq \epsilon\}$$

and the bundle E admits a "parallel frame" on B_ϵ. We can choose a hermitian structure on E so that this local parallel frame is orthogonal.

Thus if $p \in M$ is a critical point of f of index i, then we can find a local coordinate system $(x_1, \dots, x_n) : U \to \mathbb{R}^n$ centered at p such that

$$f = \frac{1}{2}(-x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2).$$

Then with respect to a Riemannian metric g on M with $g = \sum_{j=1}^n dx_j \otimes dx_j$ on $U \subset B_\epsilon$, and a local parallel frame $(s_1, \dots, s_r)$ for E, we have, for a local section $\xi = \xi_1 s_1 + \cdots + \xi_r s_r$ of $(\wedge^k T^* M) \otimes E$ on U,

$$\Delta_t^k(\xi) = (L\xi_1, \dots, L\xi_r)$$

where

$$L = \sum_{j=1}^n \left\{ - \left(\frac{\partial}{\partial x_j} \right)^2 + t^2 x_j^2 \right\} + t(-Z_1 - \cdots - Z_i + Z_{i+1} + \cdots + Z_n)$$

and

$$Z_j = [\operatorname{ext}(dx_j), \operatorname{int}(dx_j)].$$

Note that

$$Z_j(dx_{j_1} \wedge \cdots \wedge dx_{j_k}) = \begin{cases} +dx_{j_1} \wedge \cdots \wedge dx_{j_k} & \text{if } j \in \{j_1, \dots, j_k\} \\ -dx_{j_1} \wedge \cdots \wedge dx_{j_k} & \text{if } j \notin \{j_1, \dots, j_k\} \end{cases}.$$

Thus

$$(-Z_1-\cdots-Z_i+Z_{i+1}+\cdots+Z_n)(dx_{j_i}\wedge\cdots\wedge dx_{j_k}) = 2((i-\nu)+(k-\nu))-n$$

where ν is the number of elements in $\{1,\ldots,i\}\cap\{j_1,\ldots,j_k\}$.

If

$$L_{t,i,k} := \sum_{j=1}^{n}\left\{-\left(\frac{\partial}{\partial x_j}\right)^2 + t^2x_j^2\right\} + t(-Z_1-\cdots-Z_i+Z_{i+1}+\cdots+Z_n)$$

acting on k-forms on $\mathbb{R}^n$, then it is positive semi-definite and the theory of harmonic oscillator [Roe] implies that

$$\operatorname{Spec} L_{t,i,k} = t\operatorname{Spec} L_{1,i,k}$$

and the 'nullity' of $L_{1,i,k}$ is δ_{ik}. Thus

$$\lim_{t\to\infty}\operatorname{Tr} e^{-L_{t,i,k}} = \delta_{ik}.$$

Moreover, if $\rho\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\rho(0)=1$, then

$$\lim_{t\to\infty}\operatorname{Tr}(\rho e^{-L_{t,i,k}}) = \delta_{ik}.$$

Now let $\rho : M\to\mathbb{R}$ be a smooth function such that

$$\rho|_{B_{\epsilon/2}}\equiv 1, \qquad \operatorname{supp}\rho\subset B_\epsilon$$

We have, therefore,

$$\lim_{t\to\infty}\operatorname{Tr}(\rho e^{-\Delta_t^k}) = rc_{f,k}. \tag{4}$$

Finally

$$\begin{aligned}\mu_k(t) &= \int_M \operatorname{tr} K_t(x,x)\,d\operatorname{vol}(x)\\ &= \int_M \operatorname{tr}((1-\rho)K_t)\,d\operatorname{vol} + \int_M \operatorname{tr}(\rho K_t)\,d\operatorname{vol}\\ &= \int_{M-B_{\epsilon/2}} \operatorname{tr}((1-\rho)K_t)\,d\operatorname{vol} + \int_{B_\epsilon}\operatorname{tr}(\rho K_t)\,d\operatorname{vol}\end{aligned}$$

The first integral tends to 0 by the lemma kernel and the last integral $\int_{B_\epsilon}\operatorname{tr}(\rho K_t)\,d\operatorname{vol} = \operatorname{Tr}(\rho e^{-\Delta_t^k})$ tends to $rc_{f,k}$ by (4). This completes the proof.

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