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A RELATIVE MOD (H, K) NIELSEN NUMBER

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1. Introduction

Let X be a compact polyhedron, H a normal subgroup of the fundamental group $\pi_1(X)$ of X and $f : X \to X$ a selfmap such that $f_\pi H \subset H$, where $f_\pi : \pi_1(X) \to \pi_1(X)$ is the induced homomorphism by f. In the study of the fixed point theory for a map, the Nielsen number $N(f)$ gives geometric informations about the number of fixed points. However, it is not easy to compute the Nielsen number in general. In an effort to compute the Nielsen number, B.J. Jiang [4] was able to relate the Nielsen number to the mod H Nielsen number $N_H(f)$ which is a lower bound for $N(f)$.

For a pair map $f : (X, A) \to (X, A)$ of compact connected polyhedra, H. Schirmer [6] defined the relative Nielsen number $N(f; X, A)$ as a generalization of the Nielsen number. Then a natural question is, for the above pair map f, the existence of any concept of the relative Nielsen number modulo a normal subgroup as a generalization of the mod H Nielsen number. In this paper, we seek a solution of this question.

In §2, we give the defininition of the relative mod (H, K) Nielsen number $N_K^H(f; X, A)$ and show that the mod H Nielsen number $N_H(f)$ is a lower bound for the relative mod (H, K) Nielsen number $N_K^H(f; X, A)$.

In §3, we show that the relative mod (H, K) Nielsen number $N_K^H(f; X, A)$ has some basic properties, namely (the lower bound property) the homotopy invariance, the commutativity and the homotopy type invariance.

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In §4, we compute the relative mod (H, K) Nielsen number $N_K^H(f; X, A)$ under certain resrictions and give some examples.

Now let $f : X \to X$ be a selfmap of a compact polyhedron, H_1 and H_2 (not necessarily trivial) normal subgroups of $\pi_1(X)$ such that H_2 is a subgroup of H_1 and $f_\pi H_j \subset H_j$ for $j = 1, 2$. It is well known that $N_{H_2}(f) \leq N(f) = N_{\{1\}}(f)$, where $\{1\}$ is the trivial subgroup of $\pi_1(X)$. T.H. Kiang [5] showed that the inequality $N_{H_1}(f) \leq N_{H_2}(f)$ holds under certain restrictions.

In §5, we have the inequality $N_{K_1}^{H_1}(f; X, A) \leq N_{K_2}^{H_2}(f; X, A)$ which is analogous to $N_{H_1}(f) \leq N_{H_2}(f)$under certain restrictions. Finally, for two series of normal subgroups of the fundamental groups, we show that the inequality for the relative Nielsen number modulo a normal subgroup holds under certain restrictions.

2. Definitions

If $f : (X, A) \to (X, A)$ is a pair map of compact connected polyhedra, then we shall write $\bar{f} : A \to A$ for the restriction of the pair map f to A , write $f : X \to X$ if the condition that $f(A) \subset A$ is immaterial and write $i : A \to X$ for the inclusion map. The homotopies of $f : (X, A) \to (X, A)$ are maps of the form $H : (X \times I, A \times I) \to (X, A)$ and the homotopies of $f : X \to X$ are maps of the form $H : X \times I \to X$, where I is the unit interval. Throughout this paper, we always assume that all spaces are connected.

Let $i_\pi : \pi_1(A) \to \pi_1(X)$, $\bar{f}_\pi : \pi_1(A) \to \pi_1(A)$ and $f_\pi : \pi_1(X) \to \pi_1(X)$ be the induced homomorphisms. If H is a normal subgroup of $\pi_1(X)$ and K is a normal subgroup of $\pi_1(A)$ such that $i_\pi K \subset H$, $\bar{f}_\pi K \subset K$ and $f_\pi H \subset H$, then from the following commutative diagram of a morphism of selfmaps

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{f} & X, \end{array}$$

we recall the category of selfmaps with a normal subgroup and the fixed - point - class data $FPC_H(f)$ which is the weighted set of conjugacy classes of liftings $\tilde{f}_H : \widetilde{X}/H \to \widetilde{X}/H$ of f, the weight of a class $[\tilde{f}_H]$ being the index of the mod H fixed point class $p_H FPC(\tilde{f}_H)$. (See Definition 2.14 and 2.15 of Chap. III in [4].) Since $i : A \to X$ is a morphism in the category of selfmaps with a normal subgroup, from $\bar{f} : A \to A$ with K to $f : X \to X$ with H, we have a correspondence $i_{(H,K)FPC} : FPC_K(\bar{f}) \to FPC_H(f)$. (See Theorem 2.16 of Chap. III in [4].)

Thus we know the fact that every mod K fixed point class of $\bar{f} : A \to A$ is contained in a mod H fixed point class of $f : X \to X$.

In this paper, for a pair map $f : (X, A) \to (X, A)$ of compact polyhedra, we always assume that H is a normal subgroup of $\pi_1(X)$ and K is a normal subgroup of $\pi_1(A)$ such that $i_\pi K \subset H$, $\bar{f}_\pi K \subset K$ and $f_\pi H \subset H$.

Now for a pair map $f : (X, A) \to (X, A)$ of compact polyhedra, recall the definition of an essential common fixed point class of f and $\bar{f}$. (See [6].) A fixed point class F of $f : X \to X$ is a common fixed point class of f and $\bar{f}$ if F contains an essential ordinary fixed point class of $\bar{f} : A \to A$. It is called an essential common fixed point class of f and $\bar{f}$ if it is an essential ordinary fixed point class of f and common fixed point class of f and $\bar{f}$. The number of essential common fixed point classes of f and $\bar{f}$ is denoted by $N(f; \bar{f})$.

DEFINITION 2.1. Let $f : (X, A) \to (X, A)$ be a pair map of compact polyhedra. A mod H fixed point class F_H of $f : X \to X$ is a *common mod (H, K) fixed point class of f and $\bar{f}$* if F_H contains an essential mod K fixed point class of $\bar{f}$. It is an *essential common mod (H, K) fixed point class of f and $\bar{f}$* if it is an essential mod H fixed point class of f and a common mod (H, K) fixed point class of f and $\bar{f}$.

We write $N_K^H(f; \bar{f})$ for the number of essential common mod (H, K) fixed point classes of f and $\bar{f}$. Clearly $N_K^H(f; \bar{f})$ is a finite nonnegative integer.

Especially, if we choose the trivial subgroup $\{1\}$ as a normal subgroup of $\pi_1(A)$, then we have an essential common mod $(H, \{1\})$ fixed point class of f and $\bar{f}$ and we shall write the number $N_{\{1\}}^H(f; \bar{f})$ of essential common mod $(H, \{1\})$ fixed point classes of f and $\bar{f}$ for $N^H(f; \bar{f})$.

In fact, $N(f; \bar{f})$ is the number of essential common mod $(\{1\}, \{1\})$ fixed point classes of f and $\bar{f}$

REMARK 2.2. It is easy to show that $0 \leq N_K^H(f;\bar{f}) \leq N^H(f;\bar{f}) \leq N(f;\bar{f})$. If $\bar{f}_\pi K \subset J(\bar{f})$, where $J(\bar{f})$ is the Jiang subgroup of $\bar{f}: A \to A$, then any two ordinary fixed point classes of $\bar{f}: A \to A$ in a given mod K fixed point class of $\bar{f}$ have the same index (See Theorem 2.11 of Chap. III of [4].) and hence a common mod (H, K) fixed point class of f and $\bar{f}$ coincides with a common mod $(H, \{1\})$ fixed point class of f and $\bar{f}$.

For a pair map $f : (X, A) \to (X, A)$ of compact polyhedra, the relative Nielsen number $N(f; X, A)$ is defined by $N(f; X, A) = N(f) + N(\bar{f}) - N(f;\bar{f})$. (See[6].)

DEFINITION 2.3. Let $f : (X, A) \to (X, A)$ be a pair map of compact polyhedra. A *relative mod* (H, K) *Nielsen number* $N_K^H(f; X, A)$ of f (and $\bar{f}$) is defined by $N_K^H(f; X, A) = N_H(f) + N_K(\bar{f}) - N_K^H(f;\bar{f})$.
Hence $N_K^H(f; X, A)$ is a finite nonnegative integer.

Especially, if we choose the trivial subgroup $\{1\}$ as a normal subgroup of $\pi_1(A)$, then we have the relative mod $(H, \{1\})$ Nielsen number $N_{\{1\}}^H(f; X, A)$ and we denote it by $N^H(f; X, A)$.
In fact, the relative Nielsen number $N(f; X, A)$ is the relative mod $(\{1\}, \{1\})$ Nielsen number $N_{\{1\}}^{\{1\}}(f; X, A)$. (See [6].)
If $X = A$, then we have $N_K^H(f; X, A) = N_K(f)$ when K is a subgroup of H and $N^H(f; X, A) = N(f)$.
If $A = \emptyset$, then we have $N^H(f; X, A) = N_H(f)$.
If H is the trivial subgroup $\{1\}$ of $\pi_1(X)$, then we have $N_H(f) = N(f)$, $N^H(f;\bar{f}) = N(f;\bar{f})$ and hence $N^H(f; X, A) = N(f; X, A)$. Similarly if K is the trivial subgroup of $\pi_1(A)$, then we have $N_K(\bar{f}) = N(\bar{f})$, $N_K^H(f;\bar{f}) = N^H(f;\bar{f})$ and hence $N_K^H(f; X, A) = N^H(f; X, A)$.

THEOREM 2.4. *Let $f : (X, A) \to (X, A)$ be a pair map of compact polyhedra. Then we have $N_K^H(f; X, A) \leq N^H(f; X, A)$.*

Proof. Let $F_H^1, F_H^2, \cdots, F_H^l, F_H^{l+1}, \cdots, F_H^m, F_H^{m+1}, \cdots, F_H^n$ be the essential mod H fixed point classes of $f : X \to X$, where $0 < l \leq m \leq n$ are positive integers. Let $n = N_H(f)$, $l = N_K^H(f;\bar{f})$ and $m = N^H(f;\bar{f})$. And let c_i be the number of essential mod K fixed point classes of $\bar{f} : A \to A$ which are contained in F_H^i for each $1 \leq i \leq l$ and let c be the number of essential mod K fixed point classes of $\bar{f}$ which are contained in inessential mod H fixed point classes of $f : X \to X$. Similarly let d_j be the number of essential fixed point classes of $\bar{f} : A \to A$ which are contained in F_H^j for each $1 \leq j \leq m$ and let d be the number of essential fixed point classes of $\bar{f}$ which are contained in inessential mod H fixed point classes of $f : X \to X$. Then we have $N(\bar{f}) = d_1 + d_2 + \cdots + d_l + d + d_{l+1} + \cdots + d_m$. Therefore we have $N(\bar{f}) \geq c_1 + c_2 + \cdots + c_m + c + (m - l) = N_K(\bar{f}) + (m - l)$. Hence we have $N(\bar{f}) - N^H(f; \bar{f}) \geq N_K(\bar{f}) - N_K^H(f; \bar{f})$. It completes the proof.

COROLLARY 2.5. *If $f : (X, A) \to (X, A)$ is a pair map of compact polyhedra, then we have $N^H(f; X, A) \leq N(f; X, A)$.*

Proof. It is analogous to the proof of Theorem 2.4.

REMARK 2.6. It is well known that $N(f)$ is a lower bound for the relative Nielsen number $N(f; X, A)$. From the definition of the relative mod (H, K) Nielsen number $N_K^H(f; X, A)$, we obtain the fact that $N_H(f)$ is a lower bound for $N_K^H(f; X, A)$. Thus we have $N_H(f) \leq N_K^H(f; X, A) \leq N^H(f; X, A) \leq N(f; X, A)$ by Theorem 2.4 and Corollary 2.5.

THEOREM 2.7. *Let $f : (X, A) \to (X, A)$ be a pair map of compact polyhedra.*

(i) *If $N_K(\bar{f}) = 0$, then $N_K^H(f; X, A) = N_H(f)$.*

(ii) *If $N(\bar{f}) = 0$, then $N_K^H(f; X, A) = N^H(f; X, A) = N_H(f)$.*

(iii) *If $N_H(f) = 0$ or $N(f) = 0$, then $N_K^H(f; X, A) = N_K(\bar{f})$ and $N^H(f; X, A) = N(\bar{f})$.*

Proof. These are obvious from the definitions.

3. Basic properties

In this section, we show that every result about the relative Nielsen number $N(f; X, A)$ is applied to the numbers $N_K^H(f; X, A)$ and $N^H(f; X, A)$.

THEOREM 3.1. (Theorem 2.12 of Chap. III in [4]) *Let $f : X \to X$ be such that $\pi_1(X)/H$ is finite. For each lifting $\tilde{f}_H : \widetilde{X}/H \to \widetilde{X}/H$ of f, the Lefschetz number $L(\tilde{f}_H)$ of $\tilde{f}_H$ is nonzero if and only if $index(f, p_H FPC(\tilde{f}_H))$ is nonzero where $p_H : \widetilde{X}/H \to X$ is the covering map corresponding to H.*

THEOREM 3.2. (Lower bound property) *Let $f : (X, A) \to (X, A)$ be a pair map of compact polyhedra such that $\pi_1(X)/H$ and $\pi_1(A)/K$ are finite. If $L(\tilde{f}_H) \cdot L(\tilde{\bar{f}}_K) \neq 0$ for each lifting $\tilde{f}_H : \widetilde{X}/H \to \widetilde{X}/H$ of $f : X \to X$ and each lifting $\tilde{\bar{f}}_K : \widetilde{A}/K \to \widetilde{A}/K$ of $\bar{f} : A \to A$, then $f : (X, A) \to (X, A)$ has at least $N_K^H(f; X, A)$ fixed points.*

Proof. By Theorem 3.1, there exists no inessential mod $H(K)$ fixed point classes of $f : X \to X(\bar{f} : A \to A)$. Let $\bar{f} : A \to A$ have the essential mod K fixed point classes $\bar{F}_1, \bar{F}_2, \cdots, \bar{F}_l$ and $f : X \to X$ have the essential mod H fixed point classes $F_1, F_2, \cdots, F_m, F_{m+1}, \cdots, F_n$ which are indexed so that the essential common mod (H, K) fixed point classes of f and $\bar{f}$ are $F_{m+1}, F_{m+2}, \cdots, F_n$. Then

$$\begin{aligned} N_K^H(f; X, A) &= n + l - (n - m) \\ &= l + m \end{aligned}$$

Each mod K fixed point class $\bar{F}_i$ contains at least one fixed point a_i of $\bar{f}$ and each mod H fixed point class F_j contains at least one fixed point x_i of f. If $j = 1, 2, \cdots, m$, then $F_j \cap A$ is distinct from the set of essential mod K fixed point classes of $\bar{f}$. So the set $\{a_1, a_2, \cdots, a_l, x_1, x_2, \cdots, x_m\}$ consists of $l + m$ distinct points which are all fixed points of $f : (X, A) \to (X, A)$

Now throughout this section, the notation "–"(bar) on the pair map means the restriction of the pair map to the subspace of the pair of spaces.

THEOREM 3.3. (Homotopy invariance) *Let (X, A) be a pair of compact polyhedra and $i : A \to X$ be the inclusion map. Let H be a normal subgroup of $\pi_1(X)$ and K be a normal subgroup of $\pi_1(A)$ such that $i_\pi K \subset H$ where $i_\pi : \pi_1(A) \to \pi_1(X)$ is the induced homomorphism. Suppose $f_0, f_1 : (X, A) \to (X, A)$ are homotopic such that $\bar{f_0}_\pi K \subset K$, $\bar{f_1}_\pi K \subset K$, $f_{0\pi} H \subset H$ and $f_{1\pi} H \subset H$. Then we have $N_K^H(f_0; X, A) = N_K^H(f_1; X, A)$.*

Proof. It suffices to show that $N_K^H(f_0; \bar{f}_0) = N_K^H(f_1; \bar{f}_1)$. Let $G = \{g_t, \bar{g}_t\} : (X \times I, A \times I) \to (X, A)$ be a homotopy from f_0 to f_1. Let F_{0_H} be an essential common mod (H, K) fixed point class of f_0 and $\bar{f}_0$. Then F_{0_H} contains an essential mod K fixed point class $\bar{F}_{0_K}$ of $\bar{f}_0$

which corresponds to an essential mod K fixed point class $\bar{F}_{1_K}$ of $\bar{f}_1$ via $\{\bar{g}_t\}$. Thus for every $a_0 \in \bar{F}_{0_K}$ and $a_1 \in \bar{F}_{1_K}$, there exists a path $\{a_t\}$ in A such that $\langle a_t(\bar{g}_t(a_t))^{-1}\rangle \in K$. Now let F_{1_H} be the mod H fixed point class of f_1 which contains a_1. Since a path $\{a_t\}$ is in X and $\langle a_t(g_t(a_t))^{-1}\rangle \in H$ for $a_0 \in F_{0_H}$ and $a_1 \in F_{1_H}$, F_{0_H} corresponds to F_{1_H} via $\{g_t\}$. Hence F_{1_H} is an essential common mod (H,K) fixed point class of f_1 and $\bar{f}_1$ which contains $\bar{F}_{1_K}$.

Conversely, an essential common mod (H,K) fixed point class of f_1 and $\bar{f}_1$ corresponds to an essential common mod (H,K) fixed point class of f_0 and $\bar{f}_0$ via $\{g_t^{-1}\}$. Thus we have $N_K^H(f_0;\bar{f}_0) = N_K^H(f_1;\bar{f}_1)$ and hence $N_K^H(f_0;X,A) = N_K^H(f_1;X,A)$.

THEOREM 3.4. (Commutativity) *Let (X,A) and (Y,B) be compact polyhedral pairs. Let $i : A \to X$, $j : B \to Y$ be the inclusions and $f : (X,A) \to (Y,B)$ and $g : (Y,B) \to (X,A)$ be pair maps. Let H and L be normal subgroups of $\pi_1(X)$ and $\pi_1(Y)$, respectively such that $f_\pi H \subset L$ and $g_\pi L \subset H$. And let K and M be normal subgroups of $\pi_1(A)$ and $\pi_1(B)$, respectively such that $i_\pi K \subset H$, $j_\pi M \subset L$, $\bar{f}_\pi K \subset M$ and $\bar{g}_\pi M \subset K$. Then we have $N_K^H(g\circ f;X,A) = N_M^L(f\circ g;Y,B)$.*

Proof. Let F_H be an essential common mod (H,K) fixed point class of $g\circ f$ and $\bar{g}\circ\bar{f}$. Then $f(F_H)$ is an essential mod L fixed point class of $f\circ g$ and hence it suffices to show that $f(F_H)$ is a common mod (L,M) fixed point class of $f\circ g$ and $\bar{f}\circ\bar{g}$. Let $\bar{F}_K$ be an essential mod K fixed point class of $\bar{g}\circ\bar{f}$ which is contained in F_H. Since $j\circ\bar{f} = f\circ i$, $\bar{f}(\bar{F}_K)$ is contained in $f(F_H)$. Thus $f(F_H)$ is a common mod (L,M) fixed point class of $f\circ g$ and $\bar{f}\circ\bar{g}$. Similarly if G_L is an essential common mod (L,M) fixed point class of $f\circ g$ and $\bar{f}\circ\bar{g}$, then $g(G_L)$ is an essential common mod (H,K) fixed point class of $g\circ f$ and $\bar{g}\circ\bar{f}$. Thus this completes the proof.

Two maps of pairs of spaces $f : (X,A) \to (X,A)$ and $g : (Y,B) \to (Y,B)$ are said to be *maps of the same homotopy type* if there exists a homotopy equivalence $h : (X,A) \to (Y,B)$ so that the maps of pairs of spaces $h\circ f$, $g\circ h : (X,A) \to (Y,B)$ are homotopic.

THEOREM 3.5. (Homotopy type invariance) *Let (X,A) and (Y,B) be two pairs of compact polyhedra. Let $i : A \to X$, $j : B \to Y$ be the inclusions. Let H and L be normal subgroups of $\pi_1(X)$ and $\pi_1(Y)$, respectively, and let K and M be normal subgroups of $\pi_1(A)$*

and $\pi_1(B)$, respectively such that $i_\pi K \subset H$, $j_\pi M \subset L$. Suppose $f : (X, A) \to (X, A)$ and $g : (Y, B) \to (Y, B)$ are maps of the same homotopy type which have a homotopy equivalence $h : (X, A) \to (Y, B)$ such that $h_\pi H = L$, $\bar{h}_\pi K = M$, $f_\pi H \subset H$ and $\bar{f}_\pi K \subset K$. Then we have $N_K^H(f; X, A) = N_M^L(g; Y, B)$.

Proof. It is easy to see that $g_\pi L \subset L$ and $\bar{g}_\pi M \subset M$. Let $k : (Y, B) \to (X, A)$ be the homotopy inverse of h. Then

$$\begin{aligned} & N_K^H(f; X, A) \\ =& N_K^H((k \circ h)f; X, A) && (\text{ Theorem 3.3 }) \\ =& N_M^L((g \circ h)k; Y, B) && (\text{ Theorem 3.3 and Theorem 3.4 }) \\ =& N_M^L(g; Y, B) && (\text{ Theorem 3.3. }). \end{aligned}$$

4. Computations for some cases

In this section,we assume that $f : (X, A) \to (X, A)$ is a pair map of compact polyhedra and we study the relative Nielsen numbers modulo a normal subgroup for special cases and give some examples.

Now, let us consider the commutative diagram

$$\begin{array}{ccccc} \pi_1(A) & \xrightarrow{\bar{\theta}} & H_1(A) & \xrightarrow{\bar{\eta}} & Coker(1 - \bar{f}_* : H_1(A) \to H_1(A)) \\ \downarrow i_\pi & & \downarrow i_* & & \downarrow i_* \\ \pi_1(X) & \xrightarrow{\theta} & H_1(X) & \xrightarrow{\eta} & Coker(1 - f_* : H_1(X) \to H_1(X)) \end{array}$$

where $\bar{\theta}, \theta$ are abelizations and $\bar{\eta}, \eta$ are the natural projections.

Note that

$$\begin{aligned} & Coker(1 - f_* : H_1(X)/\theta(H) \to H_1(X)/\theta(H)) \\ \cong & Coker(1 - f_{1_*} : H_1(X) \to H_1(X))/\eta \circ \theta(H) \end{aligned}$$

and

$$\begin{aligned} & Coker(1 - \bar{f}_* : H_1(A)/\bar{\theta}(K) \to H_1(A)/\bar{\theta}(K)) \\ \cong & Coker(1 - \bar{f}_{1_*} : H_1(A) \to H_1(A))/\bar{\eta} \circ \bar{\theta}(K) \end{aligned}$$

THEOREM 4.1. *Let $f:(X,A)\to(X,A)$ be a pair map of compact polyhedra.*

Suppose that

(1)$\pi_1(X)/H$ and $\pi_1(A)/K$ are finite,

(2)$L(\tilde{f}_H)\cdot L(\tilde{\bar{f}}_K)\neq 0$ for each lifting $\tilde{f}_H:\widetilde{X}/H\to\widetilde{X}/H$ of $f:X\to X$ and each lifting $\tilde{\bar{f}}_K:\widetilde{A}/K\to\widetilde{A}/K$ of $\bar{f}:A\to A$ and ,

(3)$\bar{f}_\pi(\pi(A))\subset K\cdot J(\bar{f})$ and $f_\pi(\pi_1(X))\subset H\cdot J(f)$,where $J(\bar{f})$ and $J(f)$ are Jiang subgroups of $\bar{f}$ and f, respectively.

Then we have

$$\begin{aligned}N_K^H(f;X,A) &= \#Coker(1-\bar{f}_{1_*})/\bar{\eta}\circ\bar{\theta}(K)\\ &\quad+\#Coker(1-f_{1_*})/\eta\circ\theta(H)\\ &\quad-\#\{i_*Coker(1-\bar{f}_{1_*})/\bar{\eta}\circ\bar{\theta}(K)\}.\end{aligned}$$

Proof. By (1) and (2), the mod K Reidemeister number $R_K(\bar{f})$ of $\bar{f}$ equals to $N_K(\bar{f})$ and the mod H Reidemeister number $R_H(f)$ of f equals to $N_H(f)$. And by (3), we have $N_K(\bar{f})=\#Coker(1-\bar{f}_{1_*})/\bar{\eta}\circ\bar{\theta}(K)$ and $N_H(f)=\#Coker(1-f_{1_*})/\eta\circ\theta(H)$. (See Theorem 2.10 of Chap. III in [4].)

It suffices to show that $N_K^H(f;\bar{f})=\#\{i_*Coker(1-\bar{f}_{1_*})/\bar{\eta}\circ\bar{\theta}(K)\}$. The compositions $\eta\circ\theta$ and $\bar{\eta}\circ\bar{\theta}$ induce correspondences $\bar{u}:\nabla_K(\bar{f})\to Coker(1-\bar{f}_{1_*})/\bar{\eta}\circ\bar{\theta}(K)$, $u:\nabla_H(f)\to Coker(1-f_{1_*})/\eta\circ\theta(H)$ and the correspondences are bijective by (2), where $\nabla_K(\bar{f})$ is the set of $\tilde{\bar{f}}_{\pi,K}$-conjugacy classes in $\pi_1(A)/K$ and $\nabla_H(f)$ is the set of $\tilde{f}_{\pi,H}$-conjugacy classes in $\pi_1(X)/H$. (See [9].) It is easy to check that the diagram

$$\begin{array}{ccc}\nabla_K(\bar{f}) & \xrightarrow{\bar{u}} & Coker(1-\bar{f}_{1_*})/\bar{\eta}\circ\bar{\theta}(K)\\ {\scriptstyle i_\pi}\big\downarrow & & \big\downarrow{\scriptstyle i_*}\\ \nabla_H(f) & \xrightarrow{u} & Coker(1-f_{1_*})/\eta\circ\theta(H)\end{array}$$ commutes.

(See Lemma 4.5 in [10]). Let $<\alpha_H>$ be a $\tilde{f}_{\pi,H}$-conjugacy class in $\nabla_H(f)$ which corresponds to a common mod H fixed point class of f and $\bar{f}$. Then there exists a $\tilde{\bar{f}}_{\pi,K}$-conjugacy class $<\beta_K>$ in $\nabla_K(\bar{f})$

such that $i_\pi(<\beta_K>) = <\alpha_H>$. Since $u(<\alpha_H>) = u \circ i_\pi(<\beta_K>) = i_* \circ \bar{u}(<\beta_K>)$, $\bar{u}$ and u are bijective, we get $N_K^H(f;\bar{f}) = \#\{i_*Coker(1-\bar{f}_{1*})/\bar{\eta} \circ \bar{\theta}(K)\}$.

THEOREM 4.2. *Suppose X is path connected and $H = \pi_1(X)$. Then*

$$N_K^H(f;X,A) = \begin{cases} N_H(f) & \text{if } N_K(\bar{f}) = 0 \\ N_K(\bar{f}) & \text{if } N_K(\bar{f}) \neq 0. \end{cases}$$

Proof. Let x_0 and x_1 be fixed points of f. Since X is path connected, there exists a path c in X from x_0 to x_1 such that $\langle c(f \circ c)^{-1}\rangle \in \pi_1(X) = H$. By Theorem 2.2 of Chapt. III in [4], x_0 and x_1 belong to the same mod H fixed point class of $f : X \to X$. Thus $\mathrm{Fix}(f) = \{x \in X | f(x) = x\}$ is the only one mod H fixed point class of f and hence $N_H(f) \leq 1$. If $N_K(\bar{f}) = 0$, then it is obvious from Theorem 2.7. If $N_K(\bar{f}) \neq 0$, then $N_H(f) = N_K^H(f;\bar{f})$. Thus we have the conclusion.

COROLLARY 4.3. *If either X is simply connected or if f is homotopic to the identity map $id : (X,A) \to (X,A)$, then we have the same conclusion as Theorem 4.2.*

Proof. It is easy to show that $N_H(f) \leq 1$. And then apply the same arguments as in the proof of Theorem 4.2.

THEOREM 4.4. *Suppose A is path connected and $K = \pi_1(A)$. Then*

$$N_K^H(f;X,A) = \begin{cases} N_H(f)+1 & \text{if } N_K(\bar{f}) \neq 0 \text{ and } N_K^H(f;\bar{f}) = 0 \\ N_H(f) & \text{otherwise.} \end{cases}$$

Proof. Since $K = \pi_1(A)$ and A is path connected, it is easy to show that $N_K(\bar{f}) \leq 1$ by Thoerem 4.2. If $N_K(\bar{f}) = 0$, it is obvious from Theorem 2.7. Suppose $N_K(\bar{f}) \neq 0$. If $N_K^H(f;\bar{f}) \neq 0$, then $N_K^H(f;\bar{f}) = N_K(\bar{f})$ and hence $N_K^H(f;X,A) = N_H(f)$. If $N_K^H(f;\bar{f}) = 0$, then $N_K^H(f;X,A) = N_H(f) + N_K(\bar{f}) = N_H(f) + 1$.

COROLLARY 4.5. *Let the restriction $\bar{f} : A \to A$ of a compact polyhedral pair map $f : (X,A) \to (X,A)$ to A be homotopic to the identity map $id : A \to A$. Then*

$$N_K^H(f;X,A) = \begin{cases} N_H(f)+1 & \text{if } N_K(\bar{f}) \neq 0 \text{ and } N_K^H(f;\bar{f}) = 0 \\ N_H(f) & \text{otherwise.} \end{cases}$$

Proof. It is easy to show that $N_K(\bar{f}) \leq 1$. Apply the same arguments as in the proof of Theorem 4.4.

COROLLARY 4.6. *If A is simply connected, then $N_K^H(f; X, A) = N^H(f; X, A)$ and we have the same conclusion as Corollary 4.5.*

Proof. Since A is simply connected, K is the trivial group and hence $N(\bar{f}) = N_K(\bar{f}) \leq 1$. Using the same arguments as in the proof of Theorem 4.4, it completes the proof.

Now let us consider some examples.

EXAMPLE 4.7. Let $X = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 4\}$ be the annulus, $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$ and f the reflection on the x - axis. Let H be the fundamental group $\pi_1(X)$ of X and K be the fundamental group $\pi_1(A)$ of A. Clearly $\pi_1(X)$ is isomorphic to the integer group $\mathbb{Z}$, that is, $\pi_1(X) \cong \mathbb{Z}$. Since $i_\pi : \pi_1(A) \to \pi_1(X)$ is an isomorphism, $K = \pi_1(A) \cong \mathbb{Z}$.
Using Theorem 4.2, we have $N_K^H(f; X, A) = N_K(\bar{f}) = 1$ and $N^H(f; X, A) = N(\bar{f}) = 2$ and hence we have $N_K^H(f; X, A) < N^H(f; X, A)$.

EXAMPLE 4.8. Let $C = \{z = x + iy \in \mathbb{C} \mid |z - 2| = 1\}$, and $D = \{z = x + iy \in \mathbb{C} \mid |z| \leq 1\}$. Let X be their union $X = C \vee D$ with the point 1 in common and A be the figure eight which is the boundary of X. Let $f : (X, A) \to (X, A)$ be the selfmap satisfying $f(z) = (z - 2)^3 + 2$ if $z \in C$ and $f(z) = \bar{z}$ if $z \in Bd(D)$ which is the boundary circle of D. Then $\mathrm{Fix}(f) \cap A = \{-1, 1, 3\}$. We know that $\pi_1(X)$ is isomorphic to the fundamental group of the unit circle S^1, namely $\pi_1(X) \cong \mathbb{Z}$. Thus let $H = 3\mathbb{Z}$ and K be the kernel of the induced homomorphism $i_\pi : \pi_1(A) \to \pi_1(X)$. Then K is not the trivial subgroup since $\pi_1(X)$ is isomorphic to $\mathbb{Z}$ and $\pi_1(A)$ is the free group $\mathbb{Z} * \mathbb{Z}$ on two generators. It is easy to check that $i_\pi K \subset H, \bar{f}_\pi K \subset K$ and $f_\pi H \subset H$. Then we have $N_H(f) = 2$, $N_K(\bar{f}) = 1$ and $N_K^H(f; \bar{f}) = 1$ and hence we have $N_K^H(f; X, A) = 2$. By Theorem 2.4, we have $N_K^H(f; X, A) \leq N^H(f; X, A)$. In fact, $N^H(f; X, A) = 3$ and hence we have $N_K^H(f; X, A) < N^H(f; X, A)$.

EXAMPLE. 4.9. Let $X = \{z = x + iy \in \mathbb{C} \mid |z| = 1\}$, $A = 1$ and $f : (X, A) \to (X, A)$ be the reflection to x-axis. Then since we have $N(\bar{f}) = N(f; \bar{f}) = N_H(f) = 1$ and $N(f) = 2$, we have $1 = N^H(f; X, A) < N(f; X, A) = 2$. (See Corollary 2.5)

5. The relation between normal subgroups and Nielsen numbers

Let $f : X \to X$ be a selfmap on a compact polyhedron X. Let H_2 be a (not necessarily trivial group) normal subgroup of the fundamental group $\pi_1(X)$ of X such that $f_\pi H_2 \subset H_2$ where $f_\pi : \pi_1(X) \to \pi_1(X)$ is the induced homomorphism by f. It is well known that the mod H_2 Nielsen number $N_{H_2}(f)$ is a lower bound for the Nielsen number $N(f)$. Thus if we consider the Nielsen number $N(f)$ as the mod $\{1\}$ Nielsen number where $\{1\}$ is the trivial group, then we easily see that $\{1\} \prec H_2$ implies $N_{H_2}(f) \leq N_{\{1\}}(f) = N(f)$ where $A \prec B$ denotes that A is a subgroup of B.

Now let H_1 be a normal subgroup of $\pi_1(X)$ such that H_2 is a subgroup of the normal subgroup H_1 and $f_\pi H_1 \subset H_1$. We may consider quotient space $\widetilde{X}/H_j$ for $j = 1, 2$ and may obtain a commutative triangle of covering maps

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{q_2} & \widetilde{X}/H_2 \\ & & \downarrow q \\ p \searrow & & \widetilde{X}/H_1 \\ & & \downarrow p_1 \\ & & X \end{array}$$

and we will take $q : \widetilde{X}/H_2 \to \widetilde{X}/H_1$ as a covering map and $p_1 : \widetilde{X}/H_1 \to X$ and $p_2 = p_1 \circ q : \widetilde{X}/H_2 \to X$ as models of regular coverings corresponding H_1 and H_2, respectively. The group $\mathcal{D}_{H_j}$ of covering translation on this regular covering space is quotient group $\pi_1(X)/H_j$ for $j = 1, 2$.

And the lifting square

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X \end{array}$$

on which fixed point classes is based splits into three commutative squares

$$\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{X} \\
{\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle q_2} \\
\widetilde{X}/H_2 & \xrightarrow{\tilde{f}_{H_2}} & \widetilde{X}/H_2 \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\
\widetilde{X}/H_1 & \xrightarrow{\tilde{f}_{H_1}} & \widetilde{X}/H_1 \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\
X & \xrightarrow{f} & X\ ,
\end{array}$$

where $\tilde{f}_{H_j}$ is a lifting of $f$and is induced by a lifting $\tilde{f} : \widetilde{X} \to \widetilde{X}$ for $j = 1, 2$.

The only obstacle to developing a theory of fixed point classes with respect to a covering map is that not every map $\tilde{f}_{H_1} : \widetilde{X}/H_1 \to \widetilde{X}/H_1$ can be lifted to $\tilde{f}_{H_2} : \widetilde{X}/H_2 \to \widetilde{X}/H_2$. Thus we need the following lemma. (See [5].)

LEMMA 5.1. *$f_\pi H_2 \subset H_2$ if and only if such a lifting exists.*

Proof. Let $\tilde{f}_{H_{j_\pi}}$ $(j = 1, 2)$ and q_π be the induced homomorphisms. From the covering space theory, we know that such a lifting exists iff $\tilde{f}_{H_{1_\pi}} q_\pi(\pi_1(\widetilde{X}/H_2)) \subset q_\pi(\pi_1(\widetilde{X}/H_2))$. Now

$$\begin{aligned}
f_\pi H_2 \subset H_2 \text{ iff } & f_\pi p_{1_\pi} q_\pi(\pi_1(\widetilde{X}/H_2)) \subset p_{1_\pi} q_\pi(\pi_1(\widetilde{X}/H_2)) \\
& \quad (\ p_1 \circ q = p_2 \text{ is the regular covering corresponding } H_2\ ) \\
\text{iff } & p_{1_\pi} q_\pi \tilde{f}_{H_{2_\pi}}(\pi_1(\widetilde{X}/H_2)) \subset p_{1_\pi} q_\pi(\pi_1(\widetilde{X}/H_2)) \\
& \quad (\ f \circ p_1 \circ q = p_1 \circ q \circ \tilde{f}_{H_2}\ ) \\
\text{iff } & q_\pi \tilde{f}_{H_{2_\pi}}(\pi_1(\widetilde{X}/H_2)) \subset q_\pi(\pi_1(\widetilde{X}/H_2)) \\
& \quad (\ p_{1_\pi} \text{ is injective }) \\
\text{iff } & \tilde{f}_{H_{1_\pi}} q_\pi(\pi_1(\widetilde{X}/H_2)) \subset q_\pi(\pi_1(\widetilde{X}/H_2)) \\
& \quad (\ q \circ \tilde{f}_{H_2} = \tilde{f}_{H_1} \circ q\ )\ .
\end{aligned}$$

THEOREM 5.2. ([5]) *Let $f : X \to X$ be a selfmap of a compact polyhedron and let H_j $(j = 1,2)$ be a normal subgroup of $\pi_1(X)$ such that H_2 is a subgroup of H_1, $f_\pi H_j \subset H_j$ for $j = 1,2$. Then we have $N_{H_1}(f) \leq N_{H_2}(f)$.*

Now consider a pair map $f : (X, A) \to (X, A)$ of compact polyhedra. We shall write $\bar{f} : A \to A$ for the restriction of f to A and write $f : X \to X$ if the condition that $f(A) \subset A$ is immaterial. Let $i : A \to X$ be the inclusion map. Let H_j $(j = 1,2)$ be a normal subgroup of $\pi_1(X)$ such that H_2 is a subgroup of H_1 and $f_\pi H_j \subset H_j$ for $j = 1,2$. Let K_j $(j = 1,2)$ be a normal subgroup of $\pi_1(A)$ such that K_2 is a subgroup of K_1, $i_\pi K_j \subset H_j$ and $\bar{f}_\pi K_j \subset K_j$ $(j = 1,2)$ where $i_\pi : \pi_1(A) \to \pi_1(X)$ and $\bar{f}_\pi : \pi_1(A) \to \pi_1(A)$ are the induced homomorphisms. Thus as like section 2 , we can consider the relative mod (H_j, K_j) Nielsen number $N_{K_j}^{H_j}(f; X, A)$ and the relative mod $(H_j, \{1\})$ Nielsen number $N^{H_j}(f; X, A)$ for $j = 1,2$. To obtain the inequalities $N_{K_1}^{H_1}(f; X, A) \leq N_{K_2}^{H_2}(f; X, A)$ and $N^{H_1}(f; X, A) \leq N^{H_2}(f; X, A)$, we consider the following theorems.

THEOREM 5.3. *Let $f : X \to X$ be a selfmap of a compact polyhedron, let H_j $(j = 1,2)$ be a normal subgroup of $\pi_1(X)$ such that H_2 is a subgroup of H_1, $f_\pi H_j \subset H_j$ for $j = 1,2$. If $f_\pi H_1 \subset H_2 \cdot J(f)$, then any two mod H_2 fixed point classes in a given mod H_1 fixed point class have the same index.*

Proof. It is well known that $f_\pi H_1 \subset H_2 \cdot J(f)$ implies $\tilde{f}_\pi H_1 \subset H_2 \cdot J(\tilde{f})$ for any lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ of f.

Let $\tilde{f}_{H_j} : \tilde{X}/H_j \to \tilde{X}/H_j$ $(j = 1,2)$ be a lifting of f and be induced by a lifting $\tilde{f} : \tilde{X} \to \tilde{X}$. A lifting $\alpha_{H_2} \circ \tilde{f}_{H_2}$ on $\tilde{X}/H_2$ induces a lifting $\alpha_{H_1} \circ \tilde{f}_{H_1}$ on $\tilde{X}/H_1$, where α_{H_j} is the coset $H_j\alpha \in \pi_1(X)/H_j$ for $j = 1,2$. It is obvious that $\alpha_{H_1} \circ \tilde{f}_{H_1}$ is conjugate to $\tilde{f}_{H_1}$ iff $\alpha \backsim h$ for some $h \in H_1$ where $\backsim$ stands for the $\tilde{f}_{\pi,H}$ - conjugacy class. Thus if the mod H_2 fixed point class of $< \alpha_{H_2} \circ \tilde{f}_{H_2} >$ is in the same mod H_1 fixed point class as $< \tilde{f}_{H_2} >$, then $\alpha \backsim h \backsim \tilde{f}_\pi(h)$ for some $h \in H_1$, and hence $\alpha \backsim \tau_{H_2}$ where τ_{H_2} is the coset $H_2\tau \in H_2 \cdot J(\tilde{f})$. Thus

$$\begin{aligned}\operatorname{index}(f, p_2\operatorname{Fix}(\alpha_{H_2} \circ \tilde{f}_{H_2})) &= \operatorname{index}(f, p_2\operatorname{Fix}(\tau_{H_2} \circ \tilde{f}_{H_2})) \\ &= \operatorname{index}(f, p_2\operatorname{Fix}(\tilde{f}_{H_2})) \ .\end{aligned}$$

We have the conclusion.

THEOREM 5.4. *Let $f : (X, A) \to (X, A)$ be a pair map of compact polyhedra, let H_j, K_j $(j = 1, 2)$ be normal subgroups of $\pi_1(X)$, $\pi_1(A)$, respectively such that H_2 is a subgroup of H_1, K_2 is a subgroup of K_1, $i_\pi K_j \subset H_j$, $f_\pi H_j \subset H_j$ and $\bar{f}_\pi K_j \subset K_j$ for $j = 1, 2$. If $f_\pi H_1 \subset H_2 \cdot J(f)$ or $\bar{f}_\pi K_1 \subset K_2 \cdot J(\bar{f})$, then we have $N_{K_1}^{H_1}(f; X, A) \leq N_{K_2}^{H_2}(f; X, A)$.*

Proof. It suffices to show that $N_{H_1}(f) - N_{K_1}^{H_1}(f; \bar{f}) \leq N_{H_2}(f) - N_{K_2}^{H_2}(f; \bar{f})$. Let $F_{H_1}^1, F_{H_1}^2, \cdots, F_{H_1}^n$ be all essential mod H_1 fixed point classes of $f : X \to X$ and let $F_{H_1}^1, F_{H_1}^2, \cdots, F_{H_1}^m$ for $m \leq n$ be all common mod (H_1, K_1) fixed point classes of f and $\bar{f}$. And let $F_{i_1}, F_{i_2}, \cdots F_{i_{l_i}}$ be the essential mod H_2 fixed point classes of f which is contained in $F_{H_1}^i$ for each $1 \leq i \leq n$ and all $l_i \geq 1$. If $f_\pi H_1 \subset H_2 \cdot J(f)$, then we have $N_{H_2}(f) = l_1 + l_2 + \cdots + l_n$ and $N_{K_2}^{H_2}(f; \bar{f}) \leq l_1 + l_2 + \cdots + l_m$. Since $N_{K_1}^{H_1}(f; \bar{f}) = m$, we have the conclusion.

Now for the case $\bar{f}_\pi K_1 \subset K_2 \cdot J(\bar{f})$, let l' be the number of essential mod H_2 fixed point classes of f which are contained in the inessential mod H_1 fixed point classes of f. Then we have $N_{H_2}(f) = l_1 + l_2 + \cdots + l_n + l'$. and $N_{K_2}^{H_2}(f; \bar{f}) \leq l_1 + l_2 + \cdots + l_m + l'$. Thus we get the conclusion.

REMARK 5.5. (On X.) Let $f : X \to X$ be a selfmap of a compact polyhedron. Let H_j be a normal subgroup of $\pi_1(X)$ such that $f_\pi H_j \subset H_j$ for each $0 \leq j \leq n$ and also H_j be a subgroup of H_{j-1} for each $1 \leq j \leq n$. Specially, let H_0 be the fundamental group $\pi_1(X)$ and H_n the trivial group. Then we have a series $\{H_j\}$ of normal subgroups of $\pi_1(X)$, namely $\{1\} = H_n \prec H_{n-1} \prec \cdots \prec H_2 \prec H_1 \prec H_0 = \pi_1(X)$.

For each $1 \leq j \leq n-1$, we may consider quotient spaces $\widetilde{X}/H_j$, $\widetilde{X}/H_0 = X$ and $\widetilde{X}/H_n = \widetilde{X}$ and will take $q_{j+1} : \widetilde{X}/H_{j+1} \to \widetilde{X}/H_j$ as a covering map and $p_j : \widetilde{X}/H_j \to X$ as the model of the regular covering corresponding H_j. Then $p_{j+1} = p_j \circ q_{j+1}$ for each $1 \leq j \leq n-1$ and hence $p_n = p_{n-1} \circ q_n = p : \widetilde{X} \to X$ is a universal covering.

For each $2 \leq j \leq n-2$, the lifting corresponding H_j of f can be lifted to the lifting corresponding H_{j+1} of f by Lemma 5.1.

THEOREM 5.6. *Let $f : X \to X$ be a selfmap of a compact polyhedron. Suppose $\pi_1(X)$ is solvable. Then we have a series of normal subgroups satisfying conditions of Remark 5.5.*

REMARK 5.7. (On pair (X,A).) Let $f : (X, A) \to (X, A)$ be a pair map of compact polyhedra. For each $0 \leq j \leq n$, let $\{H_j\}$ be a series of normal subgroups as in Remark 5.5. Now let K_j be a normal subgroup of $\pi_1(A)$ such that $\bar{f}_\pi K_j \subset K_j$ and $i_\pi K_j \subset H_j$ for each $0 \leq j \leq n$ and also let K_j be a subgroup of K_{j-1} for each $1 \leq j \leq n$. Specially, let K_0 be the fundamental group $\pi_1(A)$ of A. Then we have a series $\{K_j\}$ of normal subgroups of $\pi_1(A)$, namely $\{1\} \prec K_n \prec K_{n-1} \cdots \prec K_2 \prec K_1 \prec K_0 = \pi_1(A)$. In fact, since $i_\pi K_n \subset H_n$, K_n is not necessarily trivial group although H_n is the trivial group. Thus we denote $\{1\}$ by K_{n+1}.

For each $1 \leq j \leq n$, as in Remark 5.5, we may consider quotient spaces $\widetilde{A}/K_j$, $\widetilde{A}/K_0 = A$ and $\widetilde{A}/K_{n+1} = \widetilde{A}$. And we will take a covering map $\bar{q}_{j+1} : \widetilde{A}/K_{j+1} \to \widetilde{A}/K_j$ and the model of the regular covering map $\bar{p}_j : \widetilde{A}/K_j \to A$ corresspoding K_j for each $1 \leq j \leq n$. Then $\bar{p}_{j+1} = \bar{p}_j \circ \bar{q}_{j+1}$ for each $1 \leq j \leq n$ and $\bar{p}_{n+1} = \bar{p}_n \circ \bar{q}_{n+1} = \bar{p} : \widetilde{A} \to A$ is a universal covering.

For each $2 \leq j \leq n-1$, the lifting corresponding K_j of $\bar{f}$ can be lifted to the lifting corresponding K_{j+1} of $\bar{f}$ by Lemma 5.1.

THEOREM 5.8. *Let $f : (X, A) \to (X, A)$ be a pair map of compact polyhedra. For each $0 \leq j \leq n$, let $\{H_j\}$ and $\{K_j\}$ be two series of normal subgroups of $\pi_1(X)$ and $\pi_1(A)$, respectively, as in Remark 5.5 and Remark 5.7. If $f_\pi H_{j-1} \subset H_j \cdot J(f)$ or $\bar{f}_\pi K_{j-1} \subset K_j \cdot J(\bar{f})$ for each $1 \leq j \leq n$,*
then we have

$$\begin{array}{ccccccccc}
N_{K_1}^{H_1}(f;X,A) & \leq & N_{K_2}^{H_2}(f;X,A) & \leq & \cdots & \leq & N_{K_{n-1}}^{H_{n-1}}(f;X,A) & & \\
\wedge| & & \wedge| & & & & \wedge| & & \\
N^{H_1}(f;X,A) & \leq & N^{H_2}(f;X,A) & \leq & \cdots & \leq & N^{H_{n-1}}(f;X,A) & \leq & N(f;X,A).
\end{array}$$

Proof. By Theorem 2.4, we have $N_{K_j}^{H_j}(f;X,A) \leq N^{H_j}(f;X,A)$ for each $1 \leq j \leq n-1$. And by Theorem 5.4, we obtain that each row holds.

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