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# CHARACTERIZATIONS OF SOME REAL HYPERSURFACES IN A COMPLEX SPACE FORM IN TERMS OF LIE DERIVATIVE*

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## 1. Introduction

A complex $n(\geq 2)$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a complex projective space $P_nC$, a complex Euclidean space $C^n$ or a complex hyperbolic space $H_nC$, according as $c > 0$, $c = 0$ or $c < 0$. Takagi [12] and Berndt [2] classified all homogeneous real hypersurfaces of $P_nC$ and $H_nC$.

Now, let $M$ be a real hypersurface of $M_n(c)$,$c\neq 0$. Then $M$ has an almost contact metric structure $(\phi, \xi, \eta, g)$ induced from the Kaehler metric and the almost complex structure of $M_n(c)$. We denote by $\mathcal{L}_\xi$ the Lie derivative with respect to $\xi$.

Recently Ki,Kim and Lee [4] gives a characterization of real hypersurfaces of type $A$. We denote by $A$ a shape operator of the real hypersurface $M$. They proved the following.

THEOREM A. *Let $M$ be a real hypersurface of $P_nC$, $n\geq 3$. If it satisfies*

$$(1.1) \qquad \mathcal{L}_\xi A = 0,$$

*where $A$ denotes the shape operator, then $M$ is locally a tube of radius $r$ over one of the following Kahler submanifolds:*

*$(A_1)$ a hyperplane $P_{n-1}C$, where $0 < r < \pi/2$,*

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$(A_2)$ *a totally geodesic* $P_kC$ $(1 < k < n-2)$,*where* $0 < r < \pi/2$.

As an example of special real hypersurfaces of $P_nC$ different from the above ones, we can give some characterizations of ruled real hypersurfaces in terms of the Lie derivative of the second fundmental form.

On the other hand, Kimura [7] obtained some properties about a ruled real hypersurface $M$ of $P_nC$,$n\geq3$. In particular, an example of minimal ruled hypersurface $M$ of $P_nC$,$n\geq3$ is constructed. Let $T_0$ be a distribution defined by a subspace $T_0(x) = \{u\in T_xM : u\perp\xi(x)\}$ of the tangent space $T_x(M)$, which is called the *holomorphic distribution.* Kimura and Maeda [8] also proved the following

THEOREM B. *Let $M$ be a real hypersurface of $P_nC$, $n\geq3$. Then the second fundamental form is $\eta$-parallel and the holomorphic distribution $T_0$ is integrable if and only if $M$ is locally a ruled real hypersurface.*

The purpose of this article is to generalize Therem A slightly and then to give another characterization of the ruled real hypersurfaces in $M_n(c)$. Assume that $\xi$ is not necessary principal. Then we can put $A\xi = \alpha\xi + \beta U$, where $U$ is a unit vector orthogonal to $\xi$ and $\alpha$ and $\xi$ are smooth functions on $M$. We prove the following.

THEOREM 1. *Let $M$ be a real hypersurface of $M_n(c)$, $c\neq0$, $n\geq3$.*

$$g((\mathcal{L}_\xi A)X, Y) = 0 \tag{1.2}$$

*for any vector fields $X$ and $Y$ in the distribution $T_0$, then $M$ is of type $A$.*

THEOREM 2. *Let $M$ be a real hypersurface of $M_n(c)$, $c\neq0$ and $n\geq3$. If it satisfies*

$$g((\mathcal{L}_\xi A)X, Y) = \beta^2 g(X, \phi U)g(Y, U) \tag{1.3}$$

*for any vector fields $X$ and $Y$ in the distribution $T_0$ and if the structure vector field is not principal and $d\alpha(\xi)\neq0$, then $M$ is locally congruent to a ruled real hypersurface.*

## 2. Preliminaries

First of all, we recall fundamental properties of real hypersurfaces of a complex space form. Let $M$ be a real hypersurface of a complex $n$-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let $C$ be a unit normal field on a neighborhood of a point $x$ in $M$. We denote by $J$ an almost complex structure of $M_n(c)$. For a local vector field $X$ on a neighborhood of $x$ in $M$, the transformation of $X$ and $C$ under $J$ can be represented as

$$JX = \phi X + \eta(X)C, \qquad JC = -\xi,$$

where $\phi$ defines a skew-symmetric transformation on the tangent bundle $TM$ of $M$, while $\eta$ and $\xi$ denote a 1-form and a vector field on a neighborhood of $x$ in $M$, respectively. Moreover, it is seen that $g(\xi, X) = \eta(X)$,where $g$ denotes the induced Riemannian metric on $M$. By properties of the almost complex structure $J$, the set $(\phi, \xi, \eta, g)$ of tensors satisfies

$$\phi^2 = -I + \eta\otimes\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where $I$ denotes the identity transformation. Accordingly, the set is so called an *almost contact metric structure*. Furthermore the covariant derivative of the structure tensors are given by

$$(2.1) \qquad (\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX,$$

where $\nabla$ is the Riemannian connection of $g$ and $A$ denotes the shape operator with respect to the unit normal $C$ on $M$.

Since the ambient space is of constant holomorphic sectional curvature $c$, the equation of Gauss and Codazzi are respectively given as follows

(2.2)
$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(2.3) \quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},$$

where $R$ denotes the Riemannian curvature tensor of $M$ and $\nabla_X A$ denotes the covariant derivative of the shape operator $A$ with respect to $X$.

The second fundamental form is said to be $\eta$-parallel if the shape operator $A$ satisfies $g((\nabla_X A)Y, Z) = 0$ for any vector fields $X$,$Y$ and $Z$ in $T_0$.

## 3. Proof of Theorem 1

Let $M$ be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geq 3$. In order to prove Theorem 1 the norm of the vector field $(A\phi-\phi A)X$ for any vector field $X$ in $T_0$ shall be estimated by the Lie derivative $\mathcal{L}_\xi A$ with respect to $\xi$. By the properties of the Lie derivative we have

$$\begin{aligned}(\mathcal{L}_\xi A)X =&\mathcal{L}_\xi(AX) - A\mathcal{L}_\xi X\\ =&(\nabla_\xi A)X - \nabla_{AX}\xi + A\nabla_X\xi.\end{aligned}$$

Consequently, by the second equation of (2.1) it is reformed to

$$\begin{aligned}(3.1)\qquad & g((\mathcal{L}_\xi A)X, Y)\\ &= g((\nabla_\xi A)X, Y) - g(\phi A^2 X, Y) + g(A\phi AX, Y),\end{aligned}$$

for any vector fields $X$ and $Y$ in $T_0$. Interchanging $X$ and $Y$ in this equation, we get

$$\begin{aligned}& g((\mathcal{L}_\xi A)Y, X)\\ &= g((\nabla_\xi A)X, Y) - g(\phi A^2 Y, X) + g(A\phi AY, X),\end{aligned}$$

from which combined with (3.1) it follows that

$$\begin{aligned}(3.2)\qquad & g((\mathcal{L}_\xi A)X, Y) - g((\mathcal{L}_\xi A)Y, X)\\ &= -g((A^2\phi - 2A\phi A + \phi A^2)X, Y).\end{aligned}$$

Now, we do not necessarily assume that $\xi$ is principal. So we put $A\xi = \alpha\xi + \beta U$, where $U$ is a unit vector field orthogonal to $\xi$ and $\alpha$ and $\beta$ are smooth functions on $M$. By the direct calculation and by using the property of the structure tensor $\phi$, the square of the norm $(A\phi - \phi A)X$ is given as follows;

$$g((A\phi - \phi A)X, (A\phi - \phi A)X) \\ = g((A^2\phi - 2A\phi A + \phi A^2)X, \phi X) - \beta^2 g(X, U)^2,$$

from which together with (3.2) it follows that

(3.3)
$$g((A\phi - \phi A)X, (A\phi - \phi A)X) \\ = g((\mathcal{L}_\xi A)\phi X, X) - g((\mathcal{L}_\xi A)X, \phi X) - \beta^2 g(X, U)^2$$

for any vector field $X$ in $T_0$.

*Proof of Theorem 1.* By the assumption $g((\mathcal{L}_\xi A)X, Y) = 0$ for any $X$ and $Y$ in $T_0$ and (3.3) it turns out to be

(3.4) $$(A\phi - \phi A)X = 0, \quad \beta g(X, U) = 0, \quad X \in T_o.$$

By the above second equation it satisfies $\beta = 0$, which means that $\xi$ is principal, i.e., $A\xi = \alpha\xi$. By this property and the first equation of (3.4) the shape operator $A$ must commute the structure tensor $\phi$. Namely we have

$$A\phi - \phi A = 0.$$

By a theorem due to Okumura [11] in $P_nC$ and Montiel and Romero [10] in $H_nC$, this completes the proof.

REMARK 1. By a theorem due to Ki,Kim and Lee [4] the following conditions are equivalent with each other:

$$(1)\mathcal{L}_\xi g = 0, \quad (2)\mathcal{L}_\xi \phi = 0, \quad (3)\mathcal{L}_\xi A = 0.$$

According to Theorem 1, it is shown that the restriction of the condition (3) to the distribution $T_0$ implies (1) and (2). However, we note that ruled real hypersurfaces satisfy

$$\mathcal{L}_\xi g(X, Y) = 0, \quad g((\mathcal{L}_\xi \phi)X, Y) = 0$$

for any vector fields $X$ and $Y$ in $T_0$, but it is not of type $A$.

REMARK 2. In the forthcoming paper ([6]) the fact that if it satisfies $(\mathcal{L}_\xi\phi)X = 0$ for any $X$ in $T_0$, then $\mathcal{L}_\xi\phi = 0$ is observed.

## 4. Ruled real hypersurfaces

This section is concerned with necessary properties about ruled real hypersurfaces. First of all, we recall a ruled real hypersurface $M$ of $M_n(c)$, $c\neq0$. Let $\gamma : I\rightarrow M_n(c)$ be any regular curve. For any $t(\in I)$ let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M = \{x\in M^{(t)}_{n-1}(c) : t\in I\}$. Then the construction of $M$ asserts that $M$ is a real hypersurface of $M_n(c)$,which is called a *ruled real hypersurface.* This means that there are many ruled real hypersurfaces of $M_n(c)$. Moreover from this construction we know that the distribution $T_0$ defined by $T_0(x) = \{X\in T_xM : X\perp\xi_x\}$ for $x\in M$ is integrable and its integral manifold is a totally geodesic submanifold $M_{n-1}(c)$ of $M_n(c)$, $c\neq0$.

Now let us give some fundamental properties of the ruled real hypersurface $M$ of $M_n(c)$,$c\neq0$. Let us put $A\xi = \alpha\xi + \beta U$, where $U$ is a unit vector orthogonal to $\xi$ and $\alpha$ and $\beta(\beta\neq0)$ are smooth functions on $M$. As is seen in [1] and [8], the shape operator $A$ satisfies

$$AU = \beta\xi. \tag{4.1}$$

Infact, if we let $AU = \beta\xi + \gamma U + \delta W$ for certain vector field $W$ orthogonal to $\xi$ and $U$, then

$$\gamma = g(AU, U) = g(-D_UC, U) = g(C, D_UU) = g(C, \nabla_UU) = 0, and$$
$$\delta = g(AU, W) = g(-D_UC, W) = g(C, D_UW) = g(C, \nabla_UW) = 0,$$

because the distribution $T_0$ is integrable and its integral manifold is totally geodesic in $M_n(c)$, $c\neq0$, where $D$ and $\nabla$ denotes the Riemannian connection of $M_n(c)$ and $M$ respectively. Moreover, from (4.1) it follows that

$$AX = 0, \tag{4.2}$$

for any vector field $X$ orthogonal to $\xi$ and $U$, because

$$g(AX,\xi) = g(A\xi, X) = g(\alpha\xi + \beta U, X) = 0,$$
$$g(AX,U) = g(AU,X) = \beta g(\xi, X) = 0, \mathit{and}$$
$$g(AX,Y) = g(-D_X C, Y) = g(C, D_X Y) = g(C, \nabla_X Y) = 0$$

for any $X$ and $Y$ in $T_0$ orthogonal to $\xi$ and $U$. Thus from (4.1) and (4.2) it turns out to be

$$A\phi X = -\beta g(X,\phi U)\xi, \quad \phi AX = 0, \quad X \in T_0, \tag{4.3}$$

which implies that

$$g((A\phi - \phi A)X, Y) = 0, X, Y \in T_0. \tag{4.4}$$

Next the covariant derivative $\nabla_\xi A$ with respect to $\xi$ is explicitly expressed. Since it satisfies

$$\begin{aligned} g((\nabla_\xi A)X, Y) &= g(\nabla_\xi(AX) - A\nabla_\xi X, Y) \\ &= g(\nabla_\xi(AX), Y) - g(\nabla_\xi X, AY), \quad X, Y \in T_0, \end{aligned}$$

we get, by the direct calculation of the left hand side of the above relation and using the property $\nabla_\xi \xi = \phi A\xi = \beta\phi U$ by (2.1),

$$g((\nabla_\xi A)X, Y) = \begin{cases} 0, & X = Y = U; \\ \beta^2 g(Y,\phi U), & X = U, Y \perp U; \\ \beta^2 g(X, \phi U), & X \perp U, Y = U; \\ 0, & X, Y \perp U. \end{cases}$$

On the other hand, we have

$$g(\phi A^2 X, Y) = \begin{cases} 0, & X = Y = U; \\ \beta^2 g(Y,\phi U), & X = U, Y \perp U; \\ 0, & X \perp U, Y = U; \\ 0, & X, Y \perp U. \end{cases}$$

Because of $\phi AX = 0$ for any $X$ in $T_0$, we get by (3.1)

$$g((\mathcal{L}_\xi A)X, Y) = \beta^2 g(X, \phi U) g(Y, U), \quad X, Y \in T_0. \tag{4.5}$$

This shows that the ruled real hypersurface $M$ of $M_n(c)$ satisfies the condition (1.3).

## 5. Proof of Theorem 2

In this section we shall give a characterization for ruled real hypersurfaces. Let $M$ be the real hypersurface of $M_n(c)$, $c\neq 0$,and assume that the structure vector is not principal. We put $A\xi = \alpha\xi + \beta U$, where $U$ is a unit vector in the holomorphic distribution $T_0$. Then by the assumption the function $\beta$ does not vanish identically on $M$.

Concerning the ruled real hypersurfaces the following is proved by the present authors [6].

THEOREM C. *Let $M$ be a real hypersurface of $M_n(c)$,$c\neq 0$ and $n\geq 3$. If it satisfies*

$$\mathcal{L}_\xi g(X,Y) = 0, \tag{5.1}$$

$$g((\mathcal{L}_\xi A\phi)X,Y) = 0 \tag{5.2}$$

*for any vector fields $X$ and $Y$ in the distribution $T_0$, and if the structure vector field is not principal and $d\alpha(\xi)\neq 0$, then $M$ is locally congruent to a ruled real hypersurface.*

Now, let $M_0$ be an open subset of $M$ consisting of points $x$ at which $\beta(x)\neq 0$. By the assumption that $\xi$ is not principal, the set $M_0$ is not empty. In order to prove the theorem,it suffices to show that the equations (5.1) and (5.2) in Theorem C hold for any vector fields $X$ and $Y$ in $T_0$ on $M_0$.

We consider first (3.3). By the assumption

$$g((\mathcal{L}_\xi A)X,Y) = \beta^2 g(X,\phi U)g(Y,U),$$

We get

$$g((A\phi-\phi A)X,(A\phi-\phi A)X) = \beta^2 g(X,\phi U)^2.$$

From this equation we can calculate the norm of $(A\phi-\phi A)X + \beta g(X, \phi U)\xi$, and we can easily obtain $(A\phi-\phi A)X + \beta g(X,\phi U)\xi = 0, X\in T_0$. This means that (5.1) holds on $M_0$.

It is also seen that (5.1) is equivalent to

$$g((\mathcal{L}_\xi \phi)X,Y) = 0, \quad X,Y\in T_0.$$

Using the property of the Lie derivative and the above equation we can prove that (5.2) holds on $M_0$. In fact, by the direct calculation, we get

$$\begin{aligned} & g((\mathcal{L}_\xi A\phi)X, Y) \\ & = g((\mathcal{L}_\xi A)\phi X, Y) + g(A(\mathcal{L}_\xi \phi)X, Y) \\ & = \beta^2 g(\phi X, \phi U)g(Y, U) + g((\mathcal{L}_\xi \phi)X, AY) \\ & = \beta^2 g(\phi X, \phi U)g(Y, U) + g((\mathcal{L}_\xi \phi)X, (AY)_0) + g(AY, \xi)g((\mathcal{L}_\xi \phi)X, \xi) \\ & = \beta^2 g(\phi X, \phi U)g(Y, U) + \beta g(Y, U)g((\mathcal{L}_\xi \phi)X, \xi), \end{aligned}$$

where $(AY)_0$ denotes the $T_0$-component of $AY$. Because of $g((\mathcal{L}_\xi \phi)X, \xi) = -\beta g(X, U)$, we can prove (5.2).

This completes the proof of Theorem 2.

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