

CONVERGENCE OF NONLINEAR ALGORITHMS

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1. Accretive operators and nonlinear semigroups

Our purpose in this paper is to prove a new version of the nonlinear Chernoff theorem and to discuss the equivalence between resolvent consistency and convergence for nonlinear algorithms acting on different Banach spaces. Such results are useful in the numerical treatment of partial differential equations via difference schemes.

Let X be a Banach space with norm $|\cdot|$. If A is a subset of $X \times X$ and $x \in X$, we let $Ax = \{y \in X : [x, y] \in A\}$. The domain of A is $D(A) = \{x \in X : Ax \neq \emptyset\}$ and the range is $R(A) = \cup\{Ax : x \in D(A)\}$. The inverse of A is defined by $A^{-1}y = \{x \in X : y \in Ax\}$.

Let ω be non-negative and let $A \subset X \times X$. The operator $A + \omega I$ is said to be accretive if

$$|(x_1 + ry_1) - (x_2 + ry_2)| \geq (1 - r\omega)|x_1 - x_2|$$

for all $[x_i, y_i] \in A$, $i = 1, 2$, and all $r > 0$.

For each $r > 0$ with $r\omega < 1$, J_r^A will denote the operator $(I + rA)^{-1}$ with $D(J_r^A) = R(I + rA)$ and $R(J_r^A) = D(A)$. It will be called the resolvent of A .

Let $A + \omega I$ be accretive. The operator $A + \omega I$ is said to be m -accretive if $R(I + rA) = X$ for all $r > 0$ with $r\omega < 1$.

Let C be a subset of X . A semigroup of type ω on C is a function $S : [0, \infty) \times C \rightarrow C$ satisfying the following conditions:

- (i) $S(0)x = x$ for $x \in C$.
- (ii) $S(t)S(s)x = S(t+s)x$ for $x \in C$ and $t, s \geq 0$.
- (iii) For each $x \in C$, $S(t)x$ is continuous in t .
- (iv) $|S(t)x - S(t)y| \leq e^{\omega t}|x - y|$ for $x, y \in C$.

Received November 5, 1993.

¹This author was partially supported by the National Science Foundation.

If $A + \omega I$ is accretive and $R(I + rA) \supset cl(D(A))$ for all sufficiently small $r > 0$, then there exists a semigroup of type ω on $cl(D(A))$ such that for each $x \in cl(D(A))$ and $t \geq 0$,

$$S(t)x = \lim_{n \rightarrow \infty} \left(I + \frac{t}{n}A\right)^{-n}x,$$

uniformly on bounded t -intervals (see [7]).

Recall that the duality mapping from X to X^* is defined by

$$J(x) = \{x^* \in X^* : (x, x^*) = |x|^2 = |x^*|^2\}.$$

It is known [9] that the accretiveness of $A + \omega I$ is equivalent to the following: for each $[x_i, y_i] \in A$, $i = 1, 2$, there exists $x^* \in J(x_1 - x_2)$ such that

$$(y_1 - y_2, x^*) \geq -\omega|x_1 - x_2|^2.$$

We now present several lemmas which will be used in the sequel. For more information on accretive operators and nonlinear semigroups, see [3, 6].

LEMMA 1.1. *Let C be a closed convex subset of X and let $T : C \rightarrow C$ be a Lipschitz continuous mapping with Lipschitz constant $\alpha \geq 1$. Then*

- (i) $(I + r(I - T))^{-1}$ exists for $0 < r < (\alpha - 1)^{-1}$
and $(I + r(I - T))^{-1} : C \rightarrow C$.
- (ii) $\rho^{-1}(I - T)$ is Lipschitz continuous
and $\rho^{-1}(I - T) + \omega I$ is accretive for $\rho > 0$,

where $\omega = \frac{1}{\rho}(\alpha - 1)$.

For a proof of Lemma 1.1, see Brezis and Pazy [4].

LEMMA 1.2. *Let T be a Lipschitz continuous (with constant $\alpha \geq 1$) self-mapping of a closed convex subset C of X . Then $T - I$ generates a semigroup $S(t)$ of type $\alpha - 1$ on C and*

$$|S(m)x - T^m x| \leq \alpha^m e^{m(\alpha-1)} \{m^2(\alpha-1)^2 + m(\alpha-1) + m\}^{\frac{1}{2}} |(I-T)x|$$

for all $x \in C$.

For a proof of Lemma 1.2, see Miyadera and Oharu [12].

The next lemma is also known (cf. [3]). We give a direct proof for completeness. In Section 3 we will need the uniform continuity of duality mappings of a sequence of Banach spaces.

LEMMA 1.3. *If X^* is a uniformly convex dual Banach space, then the duality mapping J is uniformly continuous on bounded subsets of X .*

Proof. First we will show that the duality mapping J is uniformly continuous on the unit sphere of X . Let ε be positive and let $u, v \in X$ with $|u| = 1$, $|v| = 1$ and $|u - v| < 2\delta(\varepsilon)$, where $\delta(\varepsilon)$ is the modulus of convexity of X^* . Then

$$\begin{aligned} (Ju + Jv, u) &= (Ju, u) + (Jv, v) + (Jv, u - v) \\ &= 2 + (Jv, u - v) \geq 2 - |u - v| \geq 2(1 - \delta(\varepsilon)). \end{aligned}$$

Hence $|Ju + Jv| \geq 2(1 - \delta(\varepsilon))$. Since X^* is uniformly convex, $|Ju - Jv| < \varepsilon$.

Now let B be a bounded subset of X . Then there exists $M > 1$ such that $|x| < M$ for all $x \in B$. For a given $\varepsilon > 0$, choose $0 < \delta' < \frac{\varepsilon}{2}$ such that

$$|Ju - Jv| < \frac{\varepsilon}{2M} \quad \text{if } |u| = |v| = 1 \text{ and } |u - v| < \delta'.$$

Take $\delta = \min(\varepsilon/2, (\delta')^2/2)$ and let $x, y \in B$ with $|x - y| \leq \delta$. If $|x| \leq \delta'$ and $|y| \leq \delta'$, then $|Jx| = |x| \leq \delta' < \frac{\varepsilon}{2}$ and similarly $|Jy| < \frac{\varepsilon}{2}$, and so $|Jx - Jy| < \varepsilon$. Suppose that $|x| > \delta'$ or $|y| > \delta'$, say $|x| > \delta'$. Let $u = \frac{x}{|x|}$ and $v = \frac{y}{|y|}$. Then

$$u - v = \frac{x - y}{|x|} + \left(\frac{1}{|x|} - \frac{1}{|y|} \right) y.$$

This implies that

$$|u - v| \leq \frac{|x - y|}{|x|} + \frac{||x| - |y||}{|x|} \leq 2 \frac{|x - y|}{|x|} \leq 2 \frac{|x - y|}{\delta'}.$$

Since $\delta = \min(\frac{\varepsilon}{2}, \frac{(\delta')^2}{2})$, $|u - v| < \delta'$ and so $|Ju - Jv| < 2\varepsilon/M$. Since $Jx - Jy = |x|(Ju - Jv) + (|x| - |y|)Jv$,

$$\begin{aligned} |Jx - Jy| &\leq M|Ju - Jv| + |x - y||Jv| \\ &\leq M \frac{\varepsilon}{2M} + \delta < \varepsilon. \end{aligned}$$

In Section 2 we first consider the approximation of a Banach space by a sequence of Banach spaces. We then prove our main result (Theorem 2.3) on the convergence of nonlinear algorithms. We conclude this section with an example.

In Section 3 we establish (under certain conditions) the converse of Theorem 2.3. Combining the results of Section 2 and Section 3 we obtain the equivalence between convergence and resolvent consistency for nonlinear algorithms acting on different Banach spaces.

2. The convergence of algorithms acting on different Banach spaces

In this section, a new version of the nonlinear Chernoff theorem is derived. This new version is useful in obtaining approximations of solutions to differential equations via difference schemes. This result is a nonlinear analog of the linear result given by Pazy [13] and also includes the one space linear [5] and nonlinear [4] results.

First we will consider an approximating sequence $\{X_n\}$ of Banach space. Let Z and X_n be Banach spaces with norm $|\cdot|$ and $|\cdot|_n$, $n = 1, 2, \dots$, respectively, and let X be a closed linear subspace of Z . We will make the following assumption.

ASSUMPTION.

For each $n = 1, 2, \dots$ there exist mappings $P_n : Z \rightarrow X_n$ and $E_n : X_n \rightarrow Z$ satisfying

$$(A_1) \quad |P_n x - P_n y| \leq M_1 |x - y| \text{ for all } x, y \in Z,$$

and

$$|E_n x_n - E_n y_n| \leq M_2 |x_n - y_n|_n \text{ for all } x_n, y_n \in X_n, \text{ where } M_1 \text{ and } M_2 \text{ are independent of } n;$$

$$(A_2) \quad \lim_{n \rightarrow \infty} |E_n P_n x - x| = 0 \text{ for all } x \in X;$$

$$(A_3) \quad P_n E_n x_n = x_n \text{ for all } x_n \in X_n.$$

Note that we do not assume that the spaces X_n are subspaces of X , or that the mappings P_n and E_n are bounded linear operators (cf.

[16, 17]). The introduction of X_n , P_n and E_n is motivated by the approximation of differential equations via difference equations, since the difference operators act on spaces different from the one on which the differential operator acts. A similar situation occurs in identification problems (see, for example, [1, 2]). Usually, we can choose X_n to be finite dimensional spaces if X is a function space on a bounded domain. In the following lemmas, we collect some elementary properties of P_n and E_n (see [11]).

LEMMA 2.1. *If $y_n \in X_n$, $n = 1, 2, \dots$ and $y \in X$, then*

$$\lim_{n \rightarrow \infty} |P_n y - y_n|_n = 0 \iff \lim_{n \rightarrow \infty} E_n y_n = y.$$

LEMMA 2.2. *Let $\{u_n\}$ be a sequence in Z . If $\lim_{n \rightarrow \infty} u_n = u$ and $u \in X$, then*

$$\lim_{n \rightarrow \infty} E_n P_n u_n = u.$$

Let Ω be a bounded domain in R^m . For each $n = 1, 2, \dots$ divide Ω into a finite number of disjoint sets

$$\Omega = \Omega_{n1} \cup \Omega_{n2} \cup \dots \cup \Omega_{nk(n)}$$

such that

$$\lim_{n \rightarrow \infty} \left(\max_{1 \leq j \leq k(n)} \text{diam}(\Omega_{nj}) \right) = 0.$$

Choose $x_{nj} \in \Omega_{nj}$ for each $j = 1, 2, \dots, k(n)$.

EXAMPLE 2.1. Let Ω be a bounded domain in R^m . Let Z be the space of all bounded real valued functions on $\bar{\Omega}$ with the usual supremum norm and let $X = C(\bar{\Omega})$, the subspace consisting of all continuous functions. Let $X_n = R^{k(n)}$, with the supremum norm. Define $P_n : Z \rightarrow X_n$ by

$$P_n f = (f(x_{n1}), f(x_{n2}), \dots, f(x_{nk(n)})).$$

Let $E_n : X_n \rightarrow Z$ be defined by

$$E_n u = \sum_{j=1}^{k(n)} u_j \chi_{\Omega_{nj}}.$$

Then P_n and E_n satisfy (A_1) , (A_2) and (A_3) .

Suppose that $m = 1$ and Ω is an interval. Then by using spline functions we have $E_n u \in X$ so that we can let $Z = X$. For example, if $\Omega = [0, 1]$, take, for each n , a partition of $[0, 1]$ with

$$\Delta x = \frac{1}{n} \quad \text{and } x_k = k\Delta x \text{ for } k = 1, 2, \dots, n.$$

Then define $E_n : X_n \rightarrow X$ by

$$E_n u = \sum_{i=1}^n u_k L(nx - i),$$

where

$$L(x) = \begin{cases} 1 + x & \text{if } -1 \leq x \leq 0 \\ 1 - x & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 2.2. Let Ω be a bounded domain in R^n and let each Om_{nj} be measurable and connected. Let $X = L^p(\Omega)$ with the usual L^p -norm, and let $X_n = R^{k(n)}$ with the norm $|u|_n^p = \sum_{j=1}^{k(n)} \mu(\Omega_{nj}) |u_j|^p$ where $u = (u_1, u_2, \dots, u_{k(n)})$. Define $P_n : X \rightarrow X_n$ by

$$P_n f = \left(\frac{1}{\mu(\Omega_{n1})} \int_{\Omega_{n1}} f dx, \dots, \frac{1}{\mu(\Omega_{nk(n)})} \int_{\Omega_{nk(n)}} f dx \right).$$

Let $E_n : X_n \rightarrow X$ be defined by

$$E_n u = \sum_{j=1}^{k(n)} u_j \chi_{\Omega_{nj}}.$$

Then P_n and E_n satisfy (A_1) , (A_2) and (A_3) .

THEOREM 2.3. Let $A + \omega I$ be an accretive operator in X satisfying $R(I + rA) \supset cl(D(A))$ for all $0 < r < r_0$ and let S be the semigroup

generated by $-A$. Let $\{\rho_n\}$ be a sequence of positive numbers converging to 0, and for each n let $F(\rho_n)$ be a mapping from a closed convex subset C_n of X_n into itself. Suppose that

- (1) $|F(\rho_n)x_n - F(\rho_n)y_n|_n \leq \alpha_n|x_n - y_n|_n$ for all $x_n, y_n \in C_n$, where $\alpha_n = 1 + \omega\rho_n + o(\rho_n)$,
- (2) $P_n(\text{cl}(D(A))) \subset C_n$ for each n ,
- (3) $\lim_{n \rightarrow \infty} E_n(I + \frac{r}{\rho_n}(I - F(\rho_n)))^{-1}P_nx = J_r^A x$ for all $x \in \text{cl}(D(A))$ and $0 < r < r_0$.

If $\{k_n\}$ is a sequence of integers such that $\lim_{n \rightarrow \infty} k_n\rho_n = t$, then

$$\lim_{n \rightarrow \infty} E_n F(\rho_n)^{k_n} P_n x = S(t)x \quad \text{for all } x \in \text{cl}(D(A))$$

and the convergence is uniform on bounded t -intervals.

REMARK. Suppose that $C_n = X_n$ for each n , A is m -accretive and $\{P_n\}$ is asymptotically linear (see [11]). Let $A_n = \frac{1}{\rho_n}(\alpha_n - 1)$. Then $A_n + \omega_n I$ are m -accretive operators, where $\omega_n = \frac{1}{\rho_n}(\alpha_n - 1)$. By Lemma 3.3 in [11] condition (3) in Theorem 2.3 can be replaced by the following weaker assumptions (3)':

- (3)' $\lim_{n \rightarrow \infty} E_n(I + \frac{r_1}{\rho_n}(I - F(\rho_n)))^{-1}P_nx = J_{r_1}^A x$ for all $x \in \text{cl}(D(A))$ and some $r_1 > 0$.

Proof of Theorem 2.3. Let $A_n = \rho_n^{-1}(I - F(\rho_n))$. Then by Lemma 1.1, $A_n + \omega_n I$ is an accretive operator in X_n and $R(I + rA_n) \supset \text{cl}(D(A_n))$ for $0 < r < r_0$, where $\omega_n = \frac{1}{\rho_n}(\alpha_n - 1)$. Let S_n be the semigroup generated by $-A_n$.

By assumption (3),

$$\lim_{n \rightarrow \infty} E_n J_r^{A_n} P_n x = J_r^A x \quad \text{for all } x \in \text{cl}(D(A)).$$

Hence by Theorem 3.6 in [11], we obtain

$$\lim_{n \rightarrow \infty} E_n S_n(t) P_n x = S(t)x \quad \text{for all } x \in \text{cl}(D(A))$$

and the convergence is uniform on bounded t -intervals.

Let \hat{S}_n be the semigroup generated by $-(I - F(\rho_n))$. Then

$$\begin{aligned}\hat{S}_n(t)x &= \lim_{n \rightarrow \infty} (I + \frac{t}{k}(I - F(\rho_n)))^{-k}x \\ &= \lim_{n \rightarrow \infty} (I + \frac{t\rho_n}{k} \frac{1}{\rho_n}(I - F(\rho_n)))^{-k}x = S_n(\rho_n t)x.\end{aligned}$$

Using Lemma 1.2, we obtain

$$\begin{aligned}& |S_n(k_n \rho_n) J_r^{A_n} P_n x - F(\rho_n)^{k_n} J_r^{A_n} P_n x|_n \\ &= |\hat{S}_n(k_n) J_r^{A_n} P_n x - F(\rho_n)^{k_n} J_r^{A_n} P_n x|_n \\ &\leq \alpha_n^{k_n} e^{k_n(\alpha_n - 1)} \{k_n^2(\alpha_n - 1)^2 + k_n(\alpha_n - 1) + k_n\}^{1/2} \\ &\quad |(I - F(\rho_n)) J_r^{A_n} P_n x|_n \\ &= \alpha_n^{k_n} e^{k_n(\alpha_n - 1)} \{k_n^2(\alpha_n - 1)^2 + k_n(\alpha_n - 1) + k_n\}^{1/2} \rho_n |A_n J_r^{A_n} P_n x|_n \\ &= \alpha_n^{k_n} e^{k_n(\alpha_n - 1)} \{k_n^2(\alpha_n - 1)^2 + k_n(\alpha_n - 1) + k_n\}^{1/2} \\ &\quad \rho_n \frac{1}{r} |P_n x - J_r^{A_n} P_n x|_n.\end{aligned}$$

Hence

$$\begin{aligned}& |E_n S_n(k_n \rho_n) P_n x - E_n F(\rho_n)^{k_n} P_n x| \\ &\leq |E_n S_n(k_n \rho_n) P_n x - E_n S_n(k_n \rho_n) J_r^{A_n} P_n x| \\ &\quad + |E_n S_n(k_n \rho_n) J_r^{A_n} P_n x - E_n F(\rho_n)^{k_n} J_r^{A_n} P_n x| \\ &\quad + |E_n F(\rho_n)^{k_n} J_r^{A_n} P_n x - E_n F(\rho_n)^{k_n} P_n x| \\ &\leq (K(n) + \frac{1}{r} H(n)) |P_n x - J_r^{A_n} P_n x|_n,\end{aligned}$$

where

$$K(n) = M_2(\exp(k_n \rho_n \omega_n) + \alpha_n^{k_n})$$

$$H(n) = M_2 \alpha_n^{k_n} \exp(k_n(\alpha_n - 1)) \{k_n^2(\alpha_n - 1)^2 + k_n(\alpha_n - 1) + k_n\}^{1/2} \rho_n.$$

By Lemma 2.4 in [4], we have

$$\begin{aligned}& |S_n(k_n \rho_n) J_r^{A_n} P_n - S_n(t) J_r^{A_n} P_n x|_n \\ &\leq |k_n \rho_n - t| |A_n J_r^{A_n} P_n x|_n (\exp(2\omega_n(t + k_n \rho_n)) + \exp(4\omega_n t)) \\ &\leq |k_n \rho_n - t|_n (\exp(2\omega_n(t + k_n \rho_n)) \\ &\quad + \exp(4\omega_n t)) \frac{1}{r} |P_n x - J_r^{A_n} P_n x|_n.\end{aligned}$$

Hence

$$\begin{aligned} & |E_n S_n(k_n \rho_n) P_n x - E_n S_n(t) P_n x| \\ \leq & |E_n S_n(k_n \rho_n) P_n x - E_n S_n(k_n \rho_n) J_r^{A_n} P_n x| \\ & + |E_n S_n(k_n \rho_n) J_r^{A_n} P_n x - E_n S_n(t) J_r^{A_n} P_n x| \\ & + |E_n S_n(t) J_r^{A_n} P_n x - E_n S_n(t) P_n x| \\ \leq & (L(n) + \frac{1}{r} |k_n \rho_n - t| M(n)) |P_n x - J_r^{A_n} P_n x|_n, \end{aligned}$$

where

$$\begin{aligned} L(n) &= M_2(\exp(k_n \rho_n \omega_n) + \exp(\omega_n t)) \\ M(n) &= M_2(\exp(2\omega_n(t + k_n \rho_n)) + \exp(4\omega_n t)). \end{aligned}$$

Note that

(a) If $\beta_n = o(\rho_n)$, then $\lim_{n \rightarrow \infty} k_n \beta_n = 0$.

Proof of (a). Since $k_n \rho_n \rightarrow t$ and $\beta_n / \rho_n \rightarrow 0$ as $n \rightarrow \infty$,

$$k_n \beta_n = k_n \rho_n \frac{\beta_n}{\rho_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

(b) $\lim_{n \rightarrow \infty} \omega_n = \omega$.

Proof of (b).

$$\omega_n = \frac{1}{\rho_n} (\alpha_n - 1) = \frac{1}{\rho_n} (\omega \rho_n + o(\rho_n)) \rightarrow \omega \quad \text{as } n \rightarrow \infty.$$

(c) $\lim_{n \rightarrow \infty} k_n \rho_n \omega_n = \omega t$.

Proof of (c).

$$k_n \rho_n \omega_n = k_n (\alpha_n - 1) = k_n (\omega \rho_n + o(\rho_n)) \rightarrow \omega t \quad \text{as } n \rightarrow \infty.$$

(d) $\lim_{n \rightarrow \infty} k_n (\alpha_n - 1) = \lim_{n \rightarrow \infty} k_n \rho_n \omega_n = \omega t$.

(e) $\lim_{n \rightarrow \infty} \alpha_n^{k_n} = e^{\omega t}$.

Proof of (e). Since $\alpha_n^{k_n} = (1 + \omega\rho_n + o(\rho_n))^{k_n}$ and $\lim_{n \rightarrow \infty} k_n(\omega\rho_n + o(\rho_n)) = \omega t$,

$$\lim_{n \rightarrow \infty} \alpha_n^{k_n} = e^{\omega t}.$$

Let $0 \leq t \leq T$. By (d) and (e), there exist constants Q_1 and Q_2 such that

$$H(n) \leq Q_1(Q_2 + k_n)^{1/2} \rho_n.$$

So we have $H(n) \leq C_1 k_n^{1/2} \rho_n$ for some constant C_1 . By (c) and (e), there exists a constant C_2 such that $K(n) \leq C_2$. By (b) and (c), there exist constants C_3 and C_4 such that

$$L(n) \leq C_3 \text{ and } M(n) \leq C_4.$$

Finally, for $0 \leq t \leq T$, $0 < \alpha < \frac{1}{2w}$ and $x \in D(A)$,

$$\begin{aligned} & |E_n F(\rho_n)^{k_n} P_n x - S(t)x| \\ & \leq |E_n F(\rho_n)^{k_n} P_n x - E_n S_n(k_n \rho_n) P_n x| + |E_n S_n(k_n \rho_n) P_n x \\ & \quad - E_n S_n(t) P_n x| + |E_n S_n(t) P_n x - S(t)x| \\ & \leq (C_2 + \frac{\sqrt{k_n \rho_n}}{r} C_1) |P_n x - J_r^{A_n} P_n x|_n \\ & \quad + (C_3 + \frac{1}{r} |k_n \rho_n - t| C_4) |P_n x - J_r^{A_n} P_n x|_n + |E_n S_n(t) P_n x - S(t)x| \\ & \leq (C_5 + \frac{\sqrt{k_n \rho_n}}{r} C_1 + \frac{1}{r} |k_n \rho_n - t| C_4) (2M_1 r d(0, Ax)) \\ & \quad + |J_r^{A_n} P_n x - P_n J_r^A x|_n + |E_n S_n(t) P_n x - S(t)x| \\ & \leq r(2C_5 M_1 d(0, Ax)) + (\sqrt{k_n \rho_n} C_1 + |k_n \rho_n - t| C_4) 2M_1 d(0, Ax) \\ & \quad + (C_5 + \frac{\sqrt{k_n \rho_n}}{r} C_1 + \frac{1}{r} |k_n \rho_n - t| C_4) |J_r^{A_n} P_n x - P_n J_r^A x|_n \\ & \quad + |E_n S_n(t) P_n x - S(t)x|, \end{aligned}$$

where $C_5 = C_2 + C_3$.

Let $\varepsilon > 0$ be given. Choose $r > 0$ such that

$$2rC_5 M_1 d(0, Ax) < \frac{\varepsilon}{2}.$$

Since the other terms go to zero as $n \rightarrow \infty$ for a fixed $r > 0$, the sum of all other terms is less than $\frac{\varepsilon}{2}$ for all sufficiently large n . Hence

$$\lim_{n \rightarrow \infty} E_n F(\rho_n)^{k_n} P_n x = S(t)x \quad \text{for } x \in D(A),$$

and the convergence is uniform in $t \in [0, T]$. Let $x \in cl(D(A))$, and let $\varepsilon > 0$ be given. Then there exists $y \in D(A)$ such that

$$|x - y| < \frac{\varepsilon}{2(C + e^{\alpha t})},$$

where $C > 2M_1 M_2 \exp(\omega t) + 1$ is a constant. Therefore

$$\begin{aligned} & |E_n F(\rho_n)^{k_n} P_n x - S(t)x| \\ & \leq |E_n F(\rho_n)^{k_n} P_n x - E_n F(\rho_n)^{k_n} P_n y| \\ & \quad + |E_n F(\rho_n)^{k_n} P_n y - S(t)y| + |S(t)y - S(t)x| \\ & \leq M_2 \alpha_n^{k_n} M_1 |x - y| + |E_n F(\rho_n)^{k_n} P_n y - S(t)y| + e^{\omega t} |x - y| \\ & \leq (C + e^{\alpha t}) |x - y| + |E_n F(\rho_n)^{k_n} P_n y - S(t)y| < \varepsilon \end{aligned}$$

for $0 \leq t \leq T$ and all sufficiently large n . This completes the proof.

We conclude this section by illustrating the applicability of Theorem 2.3 with the following example.

EXAMPLE 2.3. Let $X = BUC(R)$, the space of all bounded uniformly continuous functions defined on the real line R with the usual supremum norm. Let X_n, P_n and E_n be given as in Example 2.1. Let $\phi : R \rightarrow R$ be a Lipschitz continuous function with Lipschitz constant ω . Let A be an operator in X defined by

$$D(A) = \{f \in X : f, f', f'' \in X\} \quad \text{and } Af = -f''.$$

Define $B : X \rightarrow X$ by $B(f)(x) = \phi(f(x))$. Then $A + B + \omega I$ is m -accretive. Let S be the semigroup generated by $-(A + B)$.

Suppose now that $\{\rho_n\}$ is a sequence of positive numbers such that

$$1 - 2n^2 \rho_n > 0$$

for each n . We define $F(\rho_n) : X_n \rightarrow X_n$ by

$$\begin{aligned} & F(\rho_n)(\{u_k\}_{k=-\infty}^{\infty}) \\ &= \{(1 - 2n^2\rho_n)u_k + n^2\rho_n(u_{k+1} + u_{k-1}) + \rho_n\phi(u_k)\}_{k=-\infty}^{\infty}. \end{aligned}$$

Then $F(\rho_n)$ be a Lipschitz continuous function with Lipschitz constant $1 + \rho_n\omega$,

$$\lim_{n \rightarrow \infty} E_n F(\rho_n)^{k_n} P_n f = S(t)f$$

for all $f \in X$ and all sequences $\{k_n\}$ of positive integers such that $\lim_{n \rightarrow \infty} k_n \rho_n = t$, and the convergence is uniform on bounded t -intervals.

Proof. First we will show that $F(\rho_n)$ is a Lipschitz continuous function with Lipschitz constant $1 + \rho_n\omega$. Let $u, v \in X_n$. Then

$$\begin{aligned} & |F(\rho_n)u - F(\rho_n)v|_n \\ &= \sup\{|(1 - 2n^2\rho_n)(u_k - v_k) + n^2\rho_n(u_{k+1} - v_{k+1} + u_{k-1} - v_{k-1}) \\ &\quad + \rho_n(\phi(u_k) - \phi(v_k)) : 0, \pm 1, \pm 2, \dots\} \\ &\leq \sup\{(1 - 2n^2\rho_n)|u_k - v_k| + n^2\rho_n(|u_{k+1} - v_{k+1}| + |u_{k-1} - v_{k-1}|) \\ &\quad + \rho_n\omega|u_k - v_k| : k = 0, \pm 1, \pm 2, \dots\} \\ &\leq (1 - 2n^2\rho_n)|u - v|_n + n^2\rho_n(|u - v|_n + |u - v|_n) + \rho_n\omega|u - v|_n \\ &= (1 + \rho_n\omega)|u - v|_n. \end{aligned}$$

Hence $F(\rho_n)$ satisfies condition (1) of Theorem 2.3.

Since $D(F(\rho_n)) = X_n$, condition (2) of Theorem 2.3 is also satisfied. Next we will show that $A + B + \omega I$ is an m -accretive operator. It is known that A is an m -accretive operator in X . For $f, g \in X$,

$$|Bf - Bg| = \sup\{|\phi(f(x)) - \phi(g(x))| : x \in R\} \leq \omega|f - g|.$$

Thus B is a Lipschitz continuous everywhere defined function with Lipschitz constant ω , and so $B + \omega I$ is an m -accretive. Since $B + \omega I$ is continuous, everywhere defined and accretive, $A + B + \omega I$ is also an m -accretive operator in X (see [8]). By Lemma 1.1, $\frac{1}{\rho_n}(I - F(\rho_n)) + \omega I$ is an m -accretive operator. Let $A_n = \frac{1}{\rho_n}(I - F(\rho_n))$. To show that

condition (3) of Theorem 2.3 is satisfied, we will use Lemma 3.3 in [11]. Let $f \in D(A)$. Then

$$P_n(A + B)f = \{-f''(x_k) + \phi(f(x_k))\}_{k=-\infty}^{\infty}$$

and

$$\begin{aligned} A_n P_n f &= \frac{1}{\rho_n} \{f(x_k) - (1 - 2n^2 \rho_n) f(x_k) - n^2 \rho_n (f(x_{k+1}) \\ &\quad - f(x_{k-1})) + \rho_n \phi(f(x_k))\}_{k=-\infty}^{\infty} \\ &= \{n^2 (-f(x_{k+1}) + 2f(x_k) - f(x_{k-1})) + \phi(f(x_k))\}_{k=-\infty}^{\infty}. \end{aligned}$$

Thus

$$\begin{aligned} &|A_n P_n f - P_n(A + B)f| \\ &= \sup\{|n^2 (-f(x_{k+1}) + 2f(x_k) - f(x_{k-1})) + f''(x_k)| : \\ &\quad k = 0, \pm 1, \pm 2, \dots\}. \end{aligned}$$

By the uniform continuity of f'' and Lemma 2.1, we have

$$\liminf E_n A_n P_n \supset (A + B).$$

By Lemma 3.3 in [11], we can conclude that

$$\lim_{n \rightarrow \infty} E_n \left(I + \frac{r}{\rho_n} (I - F(\rho_n)) \right)^{-1} P_n f = J_r^A f,$$

since $E_n J_r^{A_n} P_n f = E_n (I + r A_n)^{-1} P_n f = E_n \left(I + \frac{r}{\rho_n} (I - F(\rho_n)) \right)^{-1} P_n f$.

Therefore we can conclude, by Theorem 2.3, that

$$\lim_{n \rightarrow \infty} E_n F(\rho_n)^{k_n} P_n f = S(t)f$$

for all $f \in X$ and all sequences $\{k_n\}$ of positive integers such that $\lim_{n \rightarrow \infty} k_n \rho_n = t$, and the convergence is uniform on bounded t -intervals.

3. Resolvent convergence

In this section we will study the converse of Theorem 2.3. The converse is not true in a general Banach space. But with some restrictions on X and X_n , we can show that convergence does imply resolvent consistency for nonlinear algorithms. Thus our sufficient condition in Section 2 turns out to be also necessary. To establish the converse, we need a linear property of $\{P_n\}$ in the following sense: $\{P_n\}$ is asymptotically linear if

$$\lim_{n \rightarrow \infty} |P_n(\alpha x + \beta y) - (\alpha P_n x + \beta P_n y)|_n = 0.$$

The asymptotic linearity of $\{P_n\}$ is equivalent to the asymptotic linearity of $\{E_n\}$. (See [11].)

To prove our result, we use Banach limits and the uniform continuity of the duality mappings on bounded subsets of X_n . Recall that a Banach limit LIM is a bounded linear functional on l^∞ of norm 1 such that

$$\liminf_{n \rightarrow \infty} x_n \leq \text{LIM } \{x_n\} \leq \limsup_{n \rightarrow \infty} x_n$$

and

$$\text{LIM } \{x_n\} = \text{LIM } \{x_{n+1}\} \quad \text{for all } \{x_n\} \in l^\infty.$$

By an argument similar to the one used in the proof of Lemma 1.3, we can prove the following lemma.

LEMMA 3.1. *Let $\{X_n^*\}$ be a sequence of uniformly convex dual Banach spaces with moduli of convexity $\delta_n(\varepsilon)$. For each n let B_n be a bounded subset of X_n , with $M_n = \sup\{|x_n|_n : x_n \in B_n\}$. Suppose that $\delta(\varepsilon) = \inf\{\delta_n(\varepsilon) : n \geq 1\}$ is positive and $M = \sup\{M_n : n \geq 1\}$ is finite. Then the duality mappings $J_n : X_n \rightarrow X_n^*$ are uniformly continuous on B_n , uniformly in n .*

EXAMPLE 3.1. Let $\{X_n\}$ be a sequence of Hilbert spaces. Since it is known that the modulus of convexity of $\delta(\varepsilon)$ of a Hilbert space is $\delta(\varepsilon) = 1 - (1 - \frac{\varepsilon^2}{4})^{1/2}$, it is clear that $\inf\{\delta_n(\varepsilon) : n \geq 1\} = 1 - (1 - \frac{\varepsilon^2}{4})^{1/2}$ is positive.

EXAMPLE 3.2. Let $X = L^p([0, 1]), p \geq 2$, and let $X_n = R^n$ with $|u|_n = \Delta x \sum_{k=1}^n |u_k|^p$. Suppose that δ_p is the modulus of convexity of X and δ_n is the modulus of convexity of X_n . Then $\delta(\varepsilon) = \inf\{\delta_p(\varepsilon), \delta_n(\varepsilon) : n \geq 1\}$ is positive.

Proof. It is well known that $\delta_p(\varepsilon) = 1 - (1 - (\varepsilon/2)^p)^{1/p}$. Let $u, v \in X_n$, and let $0 < \varepsilon < 2$. Then

$$|u_k + v_k|^p + |u_k - v_k|^p \leq 2^{p-1}(|u_k|^p + |v_k|^p) \quad \text{for each } k.$$

So we obtain

$$|u + v|_n^p + |u - v|_n^p \leq 2^{p-1}(|u|_n^p + |v|_n^p).$$

If $|u|_n \leq 1, |v|_n \leq 1$ and $|u - v|_n \geq \varepsilon$, then

$$|u + v|_n^p \leq 2^p - \varepsilon^p.$$

So we have

$$\delta_n(\varepsilon) \geq 1 - (1 - (\frac{\varepsilon}{2})^p)^{\frac{1}{p}}.$$

Using the fact that J and J_n are uniformly continuous, uniformly in n , on the bounded subsets of X and X_n , respectively, we will show that the convergence of algorithms implies resolvent consistency.

THEOREM 3.2. Let X^* and X_n^* be uniformly convex dual Banach spaces with moduli of convexity $\delta_X(\varepsilon)$ and $\delta_n(\varepsilon)$, respectively. Let $A + \omega I$ be an accretive operator in X such that $R(I + rA) \supset cl(D(A))$ for $r > 0$ with $r\omega < 1$ and let S be the semi group generated by $-A$. For each n , let C_n be a closed convex subset of X_n and let $F(\rho_n)$ be a mapping from C_n into itself such that $|F(\rho_n)x_n - F(\rho_n)y_n|_n \leq \alpha_n|x_n - y_n|_n$ with $\alpha_n = 1 + \omega\rho_n + o(\rho_n)$.

Suppose that

- (1) $\{P_n\}$ is asymptotically linear,
- (2) $\delta(\varepsilon) = \min\{\delta_X(\varepsilon), \delta_n(\varepsilon) : n \geq 1\}$ is positive,
- (3) $cl(D(A))$ is convex,
- (4) $P_n(cl(D(A))) \subset C_n$ for each n ,
- (5) $\rho_n \rightarrow 0$ as $n \rightarrow \infty$,
- (6) $\lim_{n \rightarrow \infty} E_n F(\rho_n)^{k_n} P_n x = S(t)x$ for $x \in cl(D(A))$ and any integer sequence $\{k_n\}$ with $k_n \rho_n \rightarrow t \geq 0$ as $n \rightarrow \infty$, and the convergence is uniform on bounded t -intervals.

Then $\lim_{n \rightarrow \infty} E_n(I + \frac{r}{\rho_n}(I - F(\rho_n)))^{-1} P_n x = J_r^A x$ for all $r > 0$ with $r\omega < 1$ and $x \in cl(D(A))$.

Proof. For each $T > 0$ define $\alpha(T, x) = \sup\{|E_n F(\rho_n)^k P_n x - x| : 0 \leq k \rho_n \leq T\}$. Then we claim that $\alpha(T, x) \rightarrow 0$ as $T \rightarrow 0$. If this is not so, there exist $\varepsilon > 0$ and sequences $\{k_m\}$ and $\{\rho_{n_m}\}$ such that

$$k_m \rho_{n_m} \rightarrow 0 \text{ and } |E_{n_m} F(\rho_{n_m})^{k_m} P_{n_m} x - x| \geq \varepsilon.$$

Since $k_{n_m} \rho_{n_m} \rightarrow 0$, $\lim_{n \rightarrow \infty} |E_{n_m} F(\rho_{n_m})^{k_m} P_{n_m} x - x| = 0$.

Let $y_n = (I + \frac{r}{\rho_n}(I - F(\rho_n)))^{-1} P_n x$ for $0 < r < \frac{1}{\omega}$. We will show that $\{|y_n|_n\}$ is bounded. We have

$$\begin{aligned} & \alpha_n |y_n - F(\rho_n)^k P_n x|_n \\ & \geq |F(\rho_n) y_n - F(\rho_n)^{k+1} P_n x|_n \\ & = |(1 + \frac{\rho_n}{r})(y_n - F(\rho_n)^{k+1} P_n x) + \frac{\rho_n}{r}(F(\rho_n)^{k+1} P_n x - P_n x)|_n \\ & \geq |y_n - F(\rho_n)^{k+1} P_n x|_n + \frac{\rho_n}{r} |y_n \\ & \quad - F(\rho_n)^{k+1} P_n x|_n - \frac{\rho_n}{r} |F(\rho_n)^{k+1} P_n x - P_n x|_n \\ & \geq |y_n - F(\rho_n)^{k+1} P_n x|_n + \frac{\rho_n}{r} |y_n - P_n x|_n \\ & \quad - \frac{2\rho_n}{r} |F(\rho_n)^{k+1} P_n x - P_n x|_n. \end{aligned}$$

Let $\{k_n\}$ be a sequence such that $k_n \rho_n \rightarrow t < T$ as $n \rightarrow \infty$. For $0 \leq k \leq k_n - 1$,

$$|F(\rho_n)^{k+1} P_n x - P_n x|_n \leq M_1 |E_n F(\rho_n)^{k+1} P_n x - x| \leq M_1 \alpha(T, x).$$

Hence we obtain

$$\begin{aligned} & |y_n - F(\rho_n)^k P_n x|_n \\ & \geq \alpha_n^{-1} |y_n - F(\rho_n)^{k+1} P_n x|_n + \alpha_n^{-1} \frac{\rho_n}{r} |y_n - P_n x|_n \\ & \quad - \alpha_n^{-1} \frac{2\rho_n}{r} M_1 \alpha(T, x), \end{aligned}$$

and

$$\begin{aligned} & \alpha_n^{-k} |y_n - F(\rho_n)^k P_n x|_n \\ & \geq \alpha_n^{-k-1} |y_n - F(\rho_n)^{k+1} P_n x|_n + \alpha_n^{-k-1} \frac{\rho_n}{r} |y_n - P_n x|_n \\ & \quad - \alpha_n^{-k-1} \frac{2\rho_n}{r} M_1 \alpha(T, x). \end{aligned}$$

Summing these inequities from $k = 0$ to $k = k_n - 1$, we have

$$\begin{aligned} & |y_n - P_n x|_n \\ & \geq \alpha_n^{-k_n} |y_n - F(\rho_n)^{k_n} P_n x|_n + \frac{\rho_n}{r} \left(\sum_{k=1}^{k_n} \alpha_n^{-k} \right) |y_n - P_n x|_n \\ & \quad - \frac{2\rho_n}{r} \left(\sum_{k=1}^{k_n} \alpha_n^{-k} \right) M_1 \alpha(T, x) \\ & \geq \alpha_n^{-k_n} (|y_n - P_n x|_n - |F(\rho_n)^{k_n} P_n x - P_n x|_n) \\ & \quad + \frac{\rho_n}{r} \left(\sum_{k=1}^{k_n} \alpha_n^{-k} \right) |y_n - P_n x|_n - \frac{2\rho_n}{r} \left(\sum_{k=1}^{k_n} \alpha_n^{-k} \right) M_1 \alpha(T, x). \end{aligned}$$

Hence we have

$$\left(1 - \alpha_n^{k_n} + \frac{\rho_n}{r} \frac{\alpha_n^{k_n} - 1}{\alpha_n - 1} \right) |y_n - P_n x| \leq \left(1 + \frac{2\rho_n}{r} \frac{\alpha_n^{k_n} - 1}{\alpha_n - 1} \right) M_1 \alpha(T, x),$$

and

$$\limsup_{n \rightarrow \infty} |y_n - P_n x|_n \leq \frac{1}{1 - r\omega} \left(2 + \frac{r\omega}{e^{\omega T} - 1} \right) M_1 \alpha(T, x).$$

Thus $\{|y_n|_n\}$ is bounded.

Define

$$f : cl(D(A)) \rightarrow R \quad \text{by } f(z) = \text{LIM } \{|y_n - P_n z|_n^2\},$$

where LIM is a Banach limit, $z \in cl(D(A))$ and $\{y_n\}$ is any subsequence of $\{y_n\}$ which we continue to denote by $\{y_n\}$. Then f is continuous,

convex and $f(z) \rightarrow \infty$ as $|z| \rightarrow \infty$. Since X is reflexive and $cl(D(A))$ is convex, there exists $u \in cl(D(A))$ such that

$$f(u) = \inf \{f(z) : z \in cl(D(A))\}.$$

Our next step is to show that $LIM\{(P_n z - P_n u, y_n - P_n u)\} \leq 0$ for $z \in cl(D(A))$. For $0 < \eta \leq 1$, we have

$$\begin{aligned} & (P_n z - P_n u, J_n(y_n - P_n u - \eta(P_n z - P_n u))) \\ & \leq \frac{1}{2\eta} (|y_n - P_n u|_n^2 - |y_n - P_n u - \eta(P_n z - P_n u)|_n^2). \end{aligned}$$

This implies that

$$\begin{aligned} & LIM \{(P_n z - P_n u, J_n(y_n - P_n u - \eta(P_n z - P_n u)))\} \\ & \leq \frac{1}{2\eta} (f(u) - f((1 - \eta)u + \eta z)) \leq 0. \end{aligned}$$

By the uniform continuity of J_n , uniformly in n , for each $\varepsilon > 0$ there exists η such that

$$\begin{aligned} & (P_n z - P_n u, J_n(y_n - P_n u)) \\ & \leq (P_n z - P_n u, J_n(y_n - P_n u - \eta(P_n z - P_n u))) + \varepsilon. \end{aligned}$$

Thus we have

$$\begin{aligned} & LIM \{(P_n z - P_n u, J_n(y_n - P_n u))\} \\ & \leq LIM\{(P_n z - P_n u, J_n(y_n - P_n u - \eta(P_n z - P_n u)))\} + \varepsilon \leq \varepsilon. \end{aligned}$$

Next we will show that $f(u) = 0$. Let $z \in cl(D(A))$. Then

$$\begin{aligned} & \alpha_n^2 |y_n - F(\rho_n)^k P_n z|_n^2 \\ & \geq |F(\rho_n)y_n - F(\rho_n)^{k+1} P_n z|_n^2 \\ & = |y_n - F(\rho_n)^{k+1} P_n z + \frac{\rho_n}{r}(y_n - P_n x)|_n^2 \\ & \geq |y_n - F(\rho_n)^{k+1} P_n z|_n^2 + \frac{2\rho_n}{r}(y_n - P_n x, J_n(y_n - F(\rho_n)^{k+1} P_n z)), \end{aligned}$$

and

$$\begin{aligned} & \alpha_n^{-2k} |y_n - F(\rho_n)^k P_n z|_n^2 \\ \geq & \alpha_n^{-2(k+1)} |y_n - F(\rho_n)^{k+1} P_n z|_n^2 \\ & + \alpha_n^{-2(k+1)} \frac{2\rho_n}{r} (y_n - P_n x, J_n(y_n - F(\rho_n)^{k+1} P_n z)). \end{aligned}$$

Let $\{k_n\}$ be a sequence such that $k_n \rho_n \rightarrow t < T$. Add these inequalities from $k = 0$ to $k = k_n - 1$. Then

$$\begin{aligned} & |y_n - P_n z|_n^2 \\ \geq & \alpha_n^{-2k_n} |y_n - F(\rho_n)^{k_n} P_n z|_n^2 \\ & + \frac{2\rho_n}{r} \sum_{k=1}^{k_n} \alpha_n^{-2k} (y_n - P_n x, J_n(y_n - F(\rho_n)^k P_n z)). \end{aligned}$$

Hence

$$\begin{aligned} & \frac{2\rho_n}{r} \sum_{k=1}^{k_n} \alpha_n^{-2k} (y_n - P_n x, J_n(y_n - F(\rho_n)^k P_n z)) \\ \leq & |y_n - P_n z|_n^2 - \alpha_n^{-2k_n} |y_n - F(\rho_n)^{k_n} P_n z|_n^2. \end{aligned}$$

Since $|y_n - P_n S(t)x|_n \leq |y_n - F(\rho_n)^{k_n} P_n z|_n + |F(\rho_n)^{k_n} P_n z - P_n S(t)z|_n$,

$$\begin{aligned} & \frac{2\rho_n}{r} \sum_{k=1}^{k_n} \alpha_n^{-2k} (y_n - P_n x, J_n(y_n - F(\rho_n)^k P_n z)) \\ \leq & |y_n - P_n z|_n^2 - \alpha_n^{-2k_n} |y_n - F(\rho_n)^{k_n} P_n z|_n^2 \\ \leq & |y_n - P_n z|_n^2 - \alpha_n^{-2k_n} |y_n - P_n S(t)z|_n^2 + \alpha_n^{-2k_n} M_n, \end{aligned}$$

where

$$\begin{aligned} M_n = & 2|y_n - F(\rho_n)^{k_n} P_n z|_n |F(\rho_n)^{k_n} P_n z - P_n S(t)z|_n \\ & + |F(\rho_n)^{k_n} P_n z - P_n S(t)z|_n^2. \end{aligned}$$

We apply the above to $u = z$. By the uniform continuity of J_n , uniformly in n , for each $\varepsilon > 0$ there exist $T > 0$ and n_0 such that

$$(y_n - P_n x, J_n(y_n - P_n u)) \leq \alpha_n^{-2k} (y_n - P_n x, J_n(y_n - F(\rho_n)^{k_n} P_n u)) + \varepsilon,$$

for all $n \geq n_0$ and $0 < t \leq T$. Hence

$$\begin{aligned} & \frac{2k_n \rho_n}{r} (y_n - P_n x, J_n(y_n - P_n u)) \\ & \leq \frac{\rho_n}{r} \sum_{k=1}^{k_n} \alpha_n^{-2k} (y_n - P_n x, J_n(y_n - F(\rho_n)^k P_n u)) + \frac{2k_n \rho_n}{r} \varepsilon \\ & \leq |y_n - P_n u|_n^2 - \alpha_n^{-2k_n} |y_n - P_n S(t)u|_n^2 + a n^{-2k_n} M_n + \frac{2k_n \rho_n}{r} \varepsilon. \end{aligned}$$

Apply the Banach limit LIM to both sides. Then

$$\begin{aligned} & \frac{2t}{r} \text{LIM} \{(y_n - P_n x, J_n(y_n - P_n u))\} \\ & \leq f(u) - e^{-2\omega t} f(S(t)u) + \frac{2t}{r} \varepsilon \leq (1 - e^{-2\omega t}) f(S(t)u) + \frac{2t}{r} \varepsilon. \end{aligned}$$

Hence $\text{LIM}\{(y_n - P_n x, J_n(y_n - P_n u))\} \leq \frac{r}{2t}(1 - e^{-2\omega t})f(S(t)u) + \varepsilon$.
Letting $t \rightarrow 0$, we obtain

$$\text{LIM}\{(y_n - P_n x, J_n(y_n - P_n u))\} \leq r\omega f(u) + \varepsilon.$$

Thus

$$\begin{aligned} & \text{LIM}\{(y_n - P_n x, J_n(y_n - P_n u))\} \\ & = f(u) + \text{LIM}\{(P_n u - P_n x, J_n(y_n - P_n u))\} \leq r \leq m f(u) + \varepsilon \end{aligned}$$

and

$$(1 - r\omega)f(u) \leq \varepsilon.$$

Now we can conclude that

$$f(u) = \text{LIM}\{|y_n - P_n u|_n^2\} = 0.$$

So there exists a subsequence $\{n_k\}$ such that

$$\lim_{n \rightarrow \infty} |y_{n_k} - P_{n_k} u|_{n_k} = 0.$$

Let $z_s = (I + \frac{r}{s}(I - S(s)))^{-1}$. Then $\lim_{s \rightarrow 0} z_s = J_r^A x = v$ (see [14, 15]). Suppose that $\{y_m\}$ is a subsequence of $\{y_n\}$ such that

$\lim_{m \rightarrow \infty} E_m y_m = u$. We complete the proof by showing that $u = J_r^A x$. By the uniform continuity of J_n , there exist $s_0 > 0$, $T > 0$ and n_0 such that for $s < s_0$, $t < T$, and all $n \geq n_0$

$$\begin{aligned} & \frac{2\rho_n k_n}{r} (y_n - P_n x, J_n(y_n - P_n z_s)) \\ & \leq \frac{2\rho_n}{r} \sum_{k=1}^{k_n} \alpha_n^{-2k} (y_n - P_n x, J_n(y_n - F(\rho_n)^{k_n} P_n z_s)) + \frac{2\rho_n k_n}{r} \varepsilon \\ & \leq |y_n - P_n z_s|_n^2 - \alpha_n^{-2k_n} |y_n - F(\rho_n)^{k_n} P_n z_s|_n^2 + \frac{2\rho_n k_n}{r} \varepsilon \\ & \leq |y_n - P_n z_s|_n^2 - \alpha_n^{-2k_n} |y_n - P_n S(t) z_s|_n^2 + \alpha_n^{-2k_n} N_n + \frac{2\rho_n k_n}{r} \varepsilon, \end{aligned}$$

where $N_n = 2|y_n - F(\rho_n)^{k_n} P_n z_s|_n |F(\rho_n)^{k_n} P_n z_s|_n + |F(\rho_n)^{k_n} P_n z_s - P_n S(t) z_s|_n^2$.

Choose $s = t < \min(s_0, T)$. Then

$$\begin{aligned} & \frac{2\rho_n k_n}{r} (y_n - P_n x, J_n(y_n - P_n z_t)) \\ & \leq |y_n - P_n z_t|_n^2 - \alpha_n^{-2k_n} |y_n - P_n S(t) z_t|_n^2 + \alpha_n^{-2k_n} N_n + \frac{2\rho_n k_n}{r} \varepsilon \\ & \leq |y_n - P_n z_t|_n^2 + |y_n - P_n S(t) z_t|_n^2 \\ & \quad + (1 - \alpha_n^{-2k_n}) |y_n - P_n S(t) z_t|_n^2 + \alpha_n^{-2k_n} N_n + \frac{2\rho_n k_n}{r} \varepsilon \\ & \leq |y_n - P_n z_t|_n^2 - |y_n - P_n z_t + \frac{t}{r} (P_n x - P_n z_t)|_n^2 + O_n \\ & \quad + (1 - \alpha_n^{-2k_n}) |y_n - P_n S(t) z_t|_n^2 + N_n + \frac{2\rho_n k_n}{r} \varepsilon \\ & \leq \frac{2t}{r} (P_n z_t - P_n x, J_n(y_n - P_n z_t)) + (1 - \alpha_n^{-2k_n}) |y_n - P_n S(t) z_t|_n^2 \\ & \quad + O_n + N_n + \frac{2\rho_n k_n}{r} \varepsilon, \end{aligned}$$

where

$$\begin{aligned} O_n &= 2|y_n - P_n(z_t + \frac{t}{r}(x - z_t))|_n \\ & \quad |P_n(z_t - \frac{t}{r}(x - z_t)) - (P_n z_t - \frac{t}{r}(P_n x - P_n z_t))|_n \\ & \quad + |P_n(z_t - \frac{t}{r}(x - z_t)) - P_n z_t + \frac{t}{r}(P_n x - P_n z_t)|_n^2. \end{aligned}$$

Applying LIM to both sides, we obtain

$$\begin{aligned} & \frac{2t}{r} \text{LIM}\{(y_n - P_n x, J_n(y_n - P_n z_t))\} \\ & \leq \frac{2t}{r} \text{LIM}\{(P_n z_t - P_n x, J_n(y_n - P_n z_t))\} \\ & \quad + (1 - e^{-2\omega t}) \text{LIM}\{|y_n - P_n S(t)z_t|_n^2\} + \frac{2t}{r} \varepsilon. \end{aligned}$$

Thus

$$\begin{aligned} & \text{LIM}\{(y_n - P_n x, J_n(y_n - P_n z_t))\} \\ & \leq \text{LIM}\{(P_n z_t - P_n x, J_n(y_n - P_n z_t))\} \\ & \quad + \frac{r}{2t} (1 - e^{-2\omega t}) \text{LIM}\{|y_n - P_n S(t)z_t|_n^2\} + \varepsilon \end{aligned}$$

and

$$\text{LIM}\{|y_n - P_n z_t|_n^2\} \leq \frac{r}{2t} (1 - e^{-2\omega t}) \text{LIM}\{|y_n - P_n S(t)z_t|_n^2\} + \varepsilon.$$

Letting $t \rightarrow 0$, we have

$$\text{LIM}\{|y_n - P_n v|_n^2\} \leq r\omega \text{LIM}\{|y_n - P_n v|_n^2\} + \varepsilon.$$

Therefore there exists a subsequence $\{m_k\}$ such that

$$\lim_{k \rightarrow \infty} |y_{m_k} - P_{m_k} v|_{m_k} = 0.$$

This implies that

$$\lim_{k \rightarrow \infty} |E_{m_k} y_{m_k} - v| = 0.$$

Thus $u = v$.

Combining Theorem 3.2 and Theorem 2.3, we have the following corollary concerning the equivalence of convergence and resolvent consistency.

COROLLARY 3.3. *Let X^* and X_n^* be uniformly convex Banach spaces with moduli of convexity $\delta_X(\varepsilon)$ and $\delta_n(\varepsilon)$, respectively. Let A be an accretive operator in X such that $R(I + rA) \supset cl(D(A))$ for $r > 0$ and let S be the semigroup generated by $-A$. For each n , let $F(\rho_n)$ be a nonexpansive mapping from a closed convex subset of X_n into itself.*

Suppose that

- (1) $\{P_n\}$ is asymptotically linear,
- (2) $\delta(\varepsilon) = \min\{\delta_X(\varepsilon), \delta_n(\varepsilon) : n \geq 1\}$ is positive,
- (3) $cl(D(A))$ is convex,
- (4) $P_n(cl(D(A))) \subset C_n$ for each n ,
- (5) $\rho_n \rightarrow 0$ as $n \rightarrow \infty$.

Then the following are equivalent:

- (a) $\lim_{n \rightarrow \infty} E_n(I + \frac{r}{\rho_n}(I - F(\rho_n)))^{-1} P_n x = J_r^A x$ for all $r > 0$ and $x \in cl(D(A))$.
- (b) $\lim_{n \rightarrow \infty} E_n F(\rho_n)^{k_n} P_n x = S(t)x$ for $x \in cl(D(A))$ and any integer sequence $\{k_n\}$ with $\lim_{n \rightarrow \infty} k_n \rho_n = t \geq 0$, and the convergence is uniform on bounded t -intervals.

REMARK. In the proof of Theorem 3.2 we used the uniform continuity of J_n on the bounded subsets of X_n . But we need only assume that the duality mappings are uniformly continuous from the strong topology of X_n to the weak-star topology of X_n^* . Hence the uniform convexity of X^* and X_n^* can be replaced by the following weaker condition. The duality mappings $J : X \rightarrow X^*$ and $J_n : X_n \rightarrow X_n^*$ are uniformly continuous from the strong topologies of X and X_n to the weak-star topologies of X^* and X_n^* , respectively, uniformly in n in the following sense:

For each $\varepsilon > 0$, M and $x \in X$ there exists $\delta > 0$ which depends on ε , M and x such that

$$|(P_n x, J_n u_n - J_n v_n)| < \varepsilon$$

for all $|u_n - v_n|_n < \delta$ with $|u_n|_n \leq M$ and $|v_n|_n \leq M$.

In the one space case, that is, $X = X_n$, $P_n = E_n = I$ for all n , the proofs of Theorem 3.2 and Corollary 3.3 yield the following new result (cf. Corollary 2.2 in [14]).

COROLLARY 3.4. *Let X be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let A be an accretive operator in X such that $R(I + rA) \supset cl(D(A))$ for $r > 0$ and let S be the semigroup generated by $-A$. For each n let $F(\rho_n)$ be a nonexpansive mapping from a closed convex subset of X into itself.*

Suppose that

- (1) $cl(D(A))$ is convex and $cl(D(A)) \subset C$.
- (2) $\rho_n \rightarrow 0$ as $n \rightarrow \infty$.

Then the following are equivalent:

- (a) $\lim_{n \rightarrow \infty} F(\rho_n)^{k_n} x = S(t)x$ for $x \in cl(D(A))$ and any integer sequence $\{k_n\}$ with $k_n \rho_n \rightarrow t \geq 0$ as $n \rightarrow \infty$, and the convergence is uniform on bounded t -intervals.
- (b) $\lim_{n \rightarrow \infty} (I + \frac{r}{\rho_n}(I - F(\rho_n)))^{-1} x = J_r^A x$ for all $r > 0$ and $x \in cl(D(A))$.

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