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ON THE C^r CLOSING LEMMA

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1. Introduction

Before state precisely our main theorem, we want to make some brief historical comment.

R. Thom, in 1960, was the first to consider the following problem. Can a vector field with a recurrent trajectory through a point p be perturbed so as to obtain a new vector field with a closed trajectory through p? He claimed an affirmative answer, but his argument was valid only for a C^0 small perturbation. The perception that this problem is trivial in class C^0 and very difficult in class $C^r, r \geq 1$, is due to M. Peixoto in [6]. It should be remarked that the C^r closing lemma, $r \geq 1$, in the case that M is the 2-torus T^2 and the vector field never vanishes was proved in 1962 by M. Peixoto [6] and recently by C. Gutierrez [3] for the so called constant type of vector fields on T^2 with finitely many singularities. In 1965, C. Pugh proved the C^1 closing lemma for compact manifolds of dimension two and three. In 1967, he proved the C^1 closing lemma for compact manifolds of arbitrary dimension and extend it to the case of closing a nonwandering trajectory rather than a recurrent one [7].

In 1983, C. Pugh and C. Robinson established the C^1 closing lemma when M is noncompact, provided that the point p lies on $\Omega_C = \{p \in \Omega \mid \alpha(p) \cup \omega(p) \neq \emptyset\}$.

In this paper, we prove the C^r closing lemma.

THEOREM. *Let M be a manifold, X a C^r vector field on M, and p a nonwandering point of X. Then for each $\varepsilon > 0$, there exists a*

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C^r vector field Y on M such that p is a periodic point of Y and that $\|X - Y\|_r < \varepsilon$.

2. Preliminaries

PROPOSITION 2.1. *Let M be a manifold. For each C^r vector field X on M, there exist an open subset D of $M \times \mathbb{R}$ and a C^r map $\theta : D \to M, (p,t) \mapsto \theta_t(p)$, such that*

(1) *for each $p \in M$, there exist $a(p) < 0 < b(p)$ such that $D \cap (\{p\} \times \mathbb{R}) = (a(p), b(p))$, denote $I(p)$,*
(2) *$\theta_0(p) = p$ for all $p \in M$,*
(3) *for all $p \in M$ and all $s \in I(p)$, $\frac{d}{dt}\theta_t(p)|_{t=s} = X_{\theta_s(p)}$,*
(4) *if $(p,t) \in D$, then $a(\theta_t(p)) = a(p) - t, b(\theta_t(p)) = b(p) - t$, and moreover for any $s \in I(\theta_t(p)), \theta_{s+t}(p)$ is defined and $\theta_s(\theta_t(p)) = \theta_{s+t}(p)$.*

The map θ is called the local flow generated by X. Let θ be the local flow generated by a C^r vector field X on M. For any $p \in M$, we define the first positive prolongational limit set $J^+(p)$ of p by

$$J^+(p) = \{q \in M \mid \theta_{t_n}(p_n) \to q \text{ for some sequences } p_n \to p, t_n \to \infty\}.$$

p is called a nonwandering point of X if $p \in J^+(p)$.

The following lemma is well known (cf.[2], Chapter 4, Theorem 3.14).

LEMMA 2.2. *Let X be a C^r vector field on M and let $\theta : D \to M$ be the local flow generated by X. Let p be a point of M and $X_p \neq 0$. Then there exist a C^r coordinate neighborhood (V, ψ) of $p, \nu > 0$, and a neighborhood $W \subset V$ of p such that θ restricted to $W \times (-\nu, \nu)$ is given by*

$$(x_1, \ldots, x_n, t) \mapsto (x_1 + t, x_2, \ldots, x_n).$$

In these coordinates $X = \psi_*^{-1}(\frac{\partial}{\partial x_1})$ on W.

3. Proof of the theorem

To prove the theorem, we need the following lemma.

LEMMA 3.1. *Let $\nu > 0$. For each $\varepsilon > 0$, there exists a $\delta, 0 < \delta < \nu$, such that any two points $p \in \{-\nu\} \times [-\delta, \delta]^{n-1}, q \in \{\nu\} \times [-\delta, \delta]^{n-1}$ are connected by a trajectory arc of a C^∞ vector field Y on $\mathbb{R}^n$ with*

$$||Y - \frac{\partial}{\partial x_1}||_r < \varepsilon, \quad Y = \frac{\partial}{\partial x_1} \quad \text{on} \quad \mathbb{R}^n - (-\nu, \nu)^n.$$

Proof. For $-1 \leq u_1, \dots, u_{n-1} \leq 1$, define a C^∞ vector field $Y(u_1, \dots, u_{n-1})$ on $\mathbb{R}^n$ by $Y(u_1, \dots, u_{n-1}) = \frac{\partial}{\partial x_1} + \frac{\varepsilon f}{A(r+1)\sqrt{n}} \sum_{i=1}^{n-1} u_i \frac{\partial}{\partial x_{i+1}}$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ function such that $0 < f \leq 1$ on $(-\nu, 0) \times (-\nu, \nu)^{n-1}$, $f = 0$ on $\mathbb{R}^n - (-\nu, 0) \times (-\nu, \nu)^{n-1}$, $A = \max\{||D^j f(p)|| \mid 0 \leq j \leq r, p \in \mathbb{R}^n\}$. Let $0 \leq j \leq r$. Since

$$\begin{aligned}
&\Big\| D^j \Big(Y(u_1, \dots, u_{n-1}) - \frac{\partial}{\partial x_1} \Big) \Big\| \\
&= \Big(\sum_{i=1}^{n-1} \sum_{k_1, \dots, k_j = 1}^{n} M^2 u_i^2 \Big(\frac{\partial^j f}{\partial x_{k_1} \dots \partial x_{k_j}} \Big)^2 \Big)^{\frac{1}{2}} \\
&\leq \Big(M^2 (n-1) \sum_{k_1, \dots, k_j = 1}^{n} \Big(\frac{\partial^j f}{\partial x_{k_1} \dots \partial x_{k_j}} \Big)^2 \Big)^{\frac{1}{2}} = M\sqrt{n-1}||D^j f||,
\end{aligned}$$

$$\begin{aligned}
&||Y(u_1, \dots, u_{n-1}) - \frac{\partial}{\partial x_1}||_r = \sum_{j=0}^{r} \Big\| D^j \Big(Y(u_1, \dots, u_{n-1}) - \frac{\partial}{\partial x_1} \Big) \Big\| \\
&\leq \sum_{j=0}^{r} M\sqrt{n-1}||D^j f|| \leq \sum_{j=0}^{r} M\sqrt{n-1}A = (r+1)M\sqrt{n-1}A \\
&= \frac{\sqrt{n-1}}{\sqrt{n}} \varepsilon < \varepsilon,
\end{aligned}$$

where $M = \frac{\varepsilon}{A(r+1)\sqrt{n}}$. It is clear that $Y(u_1, \dots, u_{n-1}) = \frac{\partial}{\partial x_1}$ on $\mathbb{R}^n - (-\nu, 0) \times (-\nu, \nu)^{n-1}$. Let $0 < \delta < \nu, 1 \leq i \leq n-1$. For each $p \in \{-\nu\} \times [-\delta, \delta]^{n-1}$, let p^-, p^+ be points where the trajectories of $Y(0, \dots, 0, \overset{ith}{-1}, 0, \dots, 0), Y(0, \dots, 0, \overset{ith}{1}, 0, \dots, 0)$ through p intersect

$\{0\} \times [-\nu, \nu]^{n-1}$ respectively. If $p = (-\nu, p_1, \ldots, p_{n-1})$, then there exist unique numbers $\ell_i^-(p) > 0, \ell_i^+(p) > 0$ such that

$$p^- = (0, p_1, \ldots, p_{i-1}, p_i - \ell_i^-(p), p_{i+1}, \ldots, p_{n-1}),$$
$$p^+ = (0, p_1, \ldots, p_{i-1}, p_i + \ell_i^+(p), p_{i+1}, \ldots, p_{n-1}).$$

Since $\ell_i^-(p), \ell_i^+(p)$ are continuous in p and $\{-\nu\} \times [-\delta, \delta]^{n-1}$ is compact, $\ell_i^-(p), \ell_i^+(p)$ have minimum value $d_i^-(p) > 0, d_i^+(p) > 0$ respectively. Clearly if we dimish δ, then the minimum lift $d_i^\pm(\delta)$ increases. Thus we can choose a $0 < \delta_1 < \nu$ such that $d_i^\pm(\delta_1) > 2\delta_1$ for all $1 \le i \le n-1$. Let $p = (-\nu, p_1, \ldots, p_{n-1}) \in \{-\nu\} \times [-\delta_1, \delta_1]^{n-1}, q \in \{0\} \times [-\delta_1, \delta_1]^{n-1}$. Since $\{0\} \times [-\delta_1, \delta_1]^{n-1} \subset \{0\} \times \overset{n-1}{\underset{i=1}{\pi}} [p_i - \ell_i^-(p), p_i + \ell_i^+(p)]$, there exist $-1 \le u_1, \ldots, u_{n-1} \le 1$ such that the trajectory of $Y(u_1, \ldots, u_{n-1})$ begins at p through q. Similarly, we can choose a $0 < \delta_2 < \nu$ such that any two points $p \in \{0\} \times [-\delta_2, \delta_2]^{n-1}, q \in \{\nu\} \times [-\delta_2, \delta_2]^{n-1}$ are connected by a trajectory arc of a C^∞ vector field Y on $\mathbb{R}^n$ with

$$\|Y - \frac{\partial}{\partial x_1}\|_r < \varepsilon, \quad Y = \frac{\partial}{\partial x_1} \quad \text{on} \quad \mathbb{R}^n - (0, \nu) \times (-\nu, \nu)^{n-1}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. For any two points $p \in \{-\nu\} \times [-\delta, \delta]^{n-1}, q \in \{\nu\} \times [-\delta, \delta]^{n-1}$, $p, (0, \ldots, 0)$ are connected by a trajectory arc of a C^∞ vector field Y_1 on $\mathbb{R}^n$ with

$$\|Y_1 - \frac{\partial}{\partial x_1}\|_r < \varepsilon, \quad Y_1 = \frac{\partial}{\partial x_1} \quad \text{on} \quad \mathbb{R}^n - (-\nu, 0) \times (-\nu, \nu)^{n-1}$$

and $(0, \ldots, 0), q$ are connected by a trajectory arc of a C^∞ vector field Y_2 on $\mathbb{R}^n$ with

$$\|Y_2 - \frac{\partial}{\partial x_1}\|_r < \varepsilon, \quad Y_2 = \frac{\partial}{\partial x_1} \quad \text{on} \quad \mathbb{R}^n - (0, \nu) \times (-\nu, \nu)^{n-1}.$$

Define a C^∞ vector field Y on $\mathbb{R}^n$ by $Y = Y_1 + Y_2 - \frac{\partial}{\partial x_1}$. Then, since

$$\begin{aligned} Y &= Y_1 && \text{on} \quad [-\nu, 0] \times [-\nu, \nu]^{n-1} \\ &= Y_2 && \text{on} \quad [0, \nu] \times [-\nu, \nu]^{n-1} \\ &= \frac{\partial}{\partial x_1} && \text{on} \quad \mathbb{R}^n - (-\nu, \nu)^n, \end{aligned}$$

$||Y - \frac{\partial}{\partial x_1}||_r < \varepsilon$ and the trajectory of Y beginning at p passes through $(0, \ldots, 0), q$.

We now prove the C^r closing lemma.

Proof of the Theorem. We may assume that $X_p \neq 0$. By Lemma 2.2, there exist a $\nu > 0$, C^r coordinate neighborhood (V, ψ) of p such that $\psi(p) = 0, [-\nu, \nu]^n \subset \psi(V), X = \psi_*^{-1}(\frac{\partial}{\partial x_1})$ on V. Let $A = \sup_D ||\psi_*^{-1}||$, where $D = [-\nu, \nu]^n$. Given any $\varepsilon > 0$, we can choose a $0 < \delta < \nu$ having the property described in Lemma 3.1 corresponding to $\frac{\varepsilon}{A}$. Since p is a nonwandering point of X, there exist two sequences $p_k \to p, t_k \to \infty$ such that $\theta_{t_k}(p_k) \to p$, where θ is the local flow generated by X. We can choose p_k such that $\theta_{t_k}(p_k) \in \psi^{-1}((-\nu, \nu) \times (-\delta, \delta)^{n-1})$. Let a, b be points where the trajectory of $\frac{\partial}{\partial x_1}$ through $\psi(\theta_{t_k}(p_k))$ and $\psi(p_k)$ intersects $\{-\nu\} \times [-\delta, \delta]^{n-1}, \{\nu\} \times [-\delta, \delta]^{n-1}$ respectively. It follows from the choice of δ that there exists a C^r vector field Z on $\mathbb{R}^n$ such that $||Z - \frac{\partial}{\partial x_1}||_r < \frac{\varepsilon}{A}$ and that the trajectory of Z beginning at p passes through 0 and b. Define a C^r vector field Y on M by

$$\begin{aligned} Y &= \psi_*^{-1}(Z) \quad \text{on} \quad V \\ &= X \qquad\quad\; \text{on} \quad M - V \qquad \text{(cf. [1], p207).} \end{aligned}$$

Then we have

$$\begin{aligned} ||Y - X||_r &= \left\|\psi_*^{-1}(Z) - \psi_*^{-1}\left(\frac{\partial}{\partial x_1}\right)\right\|_r \\ &= \left\|\psi_*^{-1}\left(Z - \frac{\partial}{\partial x_1}\right)\right\|_r \leq ||\psi_*^{-1}|| \; ||Z - \frac{\partial}{\partial x_1}||_r < \varepsilon. \end{aligned}$$

It is clear that p is a periodic point of Y.

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