J. Korean Math. Soc. **32** (1995), No. 1, pp. 103–108

STABILITY OF ISOMETRIES BETWEEN FINITE DIMENSIONAL HILBERT SPACES

KIL-WOUNG JUN AND DAL-WON PARK*

1. Introduction

It is a well-known classical result of Mazur and Ulam that an isometry T from a real Banach space X onto a real Banach space Y with $T(0) = 0$ is automatically linear[5]. A map T between Banach spaces X and Y is called an ϵ-bi-Lipschitz map if

$$(1-\epsilon)\|x-y\| \leq \|Tx - Ty\| \leq (1+\epsilon)\|x-y\| \text{ for } x, y \in X.$$

Jarosz[3] conjectured that if X, Y are real Banach spaces such that there is a surjective ϵ-bi-Lipschitz map between X and Y, then X and Y are linearly isomorphic for sufficiently small ϵ.

The above statement is known to be true for certain special classes of Banach spaces like uniform algebras [2]. It is also known that this is false, even for $C(K)$ spaces, if we do not assume that ϵ is close to zero[1]. Mankiewicz [4] proved that if there is a surjective ϵ-bi-Lipschitz map between a Banach space X and a Hilbert space Y, then X and Y are linearly homeomorphic. In this note we show that if T is an ϵ-bi-Lipschitz map from a Hilbert space X onto a Hilbert space Y with $\dim X < \infty$, then there is an isometry from X onto Y which is near T.

2. The result

THEOREM. *Let X and Y be real Hilbert spaces with $\dim X < \infty$. If T is an ϵ-bi-Lipschitz map from X onto Y with $T(0) = 0$ and with $\epsilon \leq \epsilon_0$, then there is an isometry I from X onto Y for which $\|Tx - Ix\| \leq$*

Received November 1, 1993.
1991 Mathematics Subject Calssification: Primary 46 E 20
Key words and Phrases: Hilbert space, ϵ-bi-Lipschitz map
* This work was partially supported by KOSEF, Grant No 91-08-00-01.

$C(\epsilon)(\|x\|^{\frac{1}{2}} + \|x\|^{\frac{3}{2}})$ *where* ϵ_0 *is an absolute constant and* $C(\epsilon) \to 0$ *as* $\epsilon \to 0$.

Proof. We divide the proof into a number of simple steps and at various points of the proof we use inequalities involving ϵ which are valid only if ϵ is sufficiently small; in these circumstances we will merely assume that ϵ is near zero. Let $e_1, e_2, ..., e_n$ be an orthonormal basis of X. We denote the inner product in X and Y by $(,)$.

STEP 1.

$$-6\sqrt{2}\epsilon - 18\epsilon^2 \leq \left(\frac{Te_i}{\|Te_i\|}, \frac{Te_j}{\|Te_j\|}\right) \leq 6\sqrt{2}\epsilon - 18\epsilon^2$$

for $i \neq j, i, j = 1, 2, ..., n$.

Proof. Let $i \neq j$ and $i, j = 1, 2, ..., n$. Since T is an ϵ-bi-Lipschitz map, $1 - \epsilon \leq \|Te_i\| \leq 1 + \epsilon$ and

$$\sqrt{2}(1-\epsilon) \leq \|Te_i - Te_j\| \leq \sqrt{2}(1+\epsilon).$$

Thus we get

$$\begin{aligned}\left\|\frac{Te_i}{\|Te_i\|} - \frac{Te_j}{\|Te_j\|}\right\| &\leq \frac{1}{\|Te_j\|}(2\epsilon + (1+\epsilon)\sqrt{2}) \\ &\leq \sqrt{2} + \frac{(2+2\sqrt{2})\epsilon}{1-\epsilon} \\ &\leq \sqrt{2} + 6\epsilon.\end{aligned}$$

Also, we have

$$\begin{aligned}\left\|\frac{Te_i}{\|Te_i\|} - \frac{Te_j}{\|Te_j\|}\right\| &\geq \frac{1}{\|Te_j\|}\Big|\, \|Te_i - Te_j\| - \big|\, \|Te_j\| - \|Te_i\|\,\big|\,\Big| \\ &\geq \frac{1}{1+\epsilon}((1-\epsilon)\sqrt{2} - 2\epsilon) \\ &\geq \sqrt{2} - 6\epsilon.\end{aligned}$$

By the above two inequalities, we have

$$\sqrt{2} - 6\epsilon \leq \left\|\frac{Te_i}{\|Te_i\|} - \frac{Te_j}{\|Te_j\|}\right\| \leq \sqrt{2} + 6\epsilon.$$

Hence

$$-6\sqrt{2}\epsilon - 18\epsilon^2 \leq \left(\frac{Te_i}{\|Te_i\|}, \frac{Te_j}{\|Te_j\|}\right) \leq 6\sqrt{2}\epsilon - 18\epsilon^2.$$

STEP 2. There is an orthonormal basis $f_1, f_2, ..., f_n$ of Y for which

$$\|f_i - \frac{Te_i}{\|Te_i\|}\| \leq C_i(\epsilon), i = 1, 2, ..., n$$

where $C_i(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof. Put $f_1 = \frac{Te_1}{\|Te_1\|}$. Then $C_1(\epsilon) = 0$. Suppose $f_1, ..., f_m (m < n)$ are linearly independent such that $\|f_i\| = 1, (f_i, f_j) = 0$ for $i \neq j, i, j = 1, ..., m$ and

$$\|f_i - \frac{Te_i}{\|Te_i\|}\| \leq C_i(\epsilon), i = 1, 2. ..., m.$$

Let $Se_{m+1} = \{f \in Y | \|f\| = 1, (f, f_1) = \cdots = (f, f_m) = 0, \frac{Te_{m+1}}{\|Te_{m+1}\|} = \alpha_1 f_1 + \cdots + \alpha_m f_m + \beta f,\quad \alpha_1, \alpha_2, ..., \alpha_m, \beta$ are real numbers $\}$. Suppose that Se_{m+1} is empty. Without loss of generality we can assume that $\frac{Te_{m+1}}{\|Te_{m+1}\|} = \alpha_1 f_1 + \cdots + \alpha_m f_m$. Then, by Schwarz inequality and Step 1,

$$\left|\left(\frac{Te_{m+1}}{\|Te_{m+1}\|}, f_j\right)\right| \leq 6\sqrt{2}\epsilon + 18\epsilon^2 + C_j(\epsilon).$$

Thus we have

$$\begin{aligned}\left(\frac{Te_{m+1}}{\|Te_{m+1}\|}, \frac{Te_{m+1}}{\|Te_{m+1}\|}\right) &\leq (|\alpha_1| + \cdots + |\alpha_m|)(6\sqrt{2}\epsilon + 18\epsilon^2 + C_1(\epsilon) \\ &\quad + \cdots + C_m(\epsilon)) \\ &< m(6\sqrt{2}\epsilon + 18\epsilon^2 + C_1(\epsilon) + \cdots + C_m(\epsilon)) \\ &< 1.\end{aligned}$$

This contradicts that $\|\frac{Te_{m+1}}{\|Te_{m+1}\|}\| = 1$. Thus $Se_{m+1} \neq \phi$. We choose a $f \in Se_{m+1}$ and let $f_{m+1} = f$. Thus $\dim X \leq \dim Y$. Since T

is an ϵ-bi-Lipschitz map, $\dim Y \leq \dim X$. Hence $f_1, f_2, ..., f_n$ is an orthonormal basis of Y. Since $\frac{Te_{m+1}}{\|Te_{m+1}\|} = \alpha_1 f_1 + \cdots + \alpha_m f_m + \beta f_{m+1}$,

$$\left\|f_{m+1} - \frac{Te_{m+1}}{\|Te_{m+1}\|}\right\|^2 = 2 - 2\beta.$$

Since $(\frac{Te_{m+1}}{\|Te_{m+1}\|}, f_i) = \alpha_i, i = 1, 2, ..., m$, we have $|\alpha_i| \leq 6\sqrt{2}\epsilon + 18\epsilon^2 + C_i(\epsilon)$. Thus

$$\begin{aligned}\beta^2 &= 1 - \alpha_1^2 - \alpha_2^2 - \cdots - \alpha_m^2 \\ &\geq 1 - m(6\sqrt{2}\epsilon + 18\epsilon^2) - (C_1(\epsilon) + C_2(\epsilon) + \cdots + C_m(\epsilon)).\end{aligned}$$

Hence

$$\begin{aligned}2 - 2\beta &\leq 2 - 2\beta^2 \\ &\leq 2m(6\sqrt{2}\epsilon + 18\epsilon^2) + 2(C_1(\epsilon) + C_2(\epsilon) + \cdots + C_m(\epsilon)).\end{aligned}$$

Let $C_{m+1}(\epsilon) = \sqrt{2m(6\sqrt{2}\epsilon + 18\epsilon^2) + 2(C_1(\epsilon) + C_2(\epsilon) + \cdots + C_m(\epsilon))}$. Then $C_{m+1}(\epsilon) \to 0$ as $\epsilon \to 0$.

STEP 3. $\|\lambda Tx - T\lambda x\| \leq 4\sqrt{\epsilon}(|\lambda|^{\frac{1}{2}} + |\lambda|^{\frac{3}{2}})\|x\|$ for $\lambda \in R$ and $\|Tx + Ty - T(x+y)\| \leq 4\sqrt{\epsilon}(\|x\| + \|y\|)$.

Proof. Since T is an ϵ-bi-Lipschitz map,

$$(1-\epsilon)|1-\lambda|\|x\| \leq \|Tx - T\lambda x\| \leq (1+\epsilon)|1-\lambda|\|x\|.$$

A routine calculation shows that

$$\begin{aligned}(1-\epsilon)^2(1-\lambda)^2\|x\|^2 - \|Tx\|^2 - \|T\lambda x\|^2 \\ &\leq -2(Tx, T\lambda x) \\ &\leq (1+\epsilon)^2(1-\lambda)^2\|x\|^2 - \|Tx\|^2 - \|T\lambda x\|^2.\end{aligned}$$

So we have

$$\|\lambda Tx - T\lambda x\| \leq 4|\lambda|^{\frac{3}{2}}\sqrt{\epsilon}\|x\| \text{ for } \lambda \geq 1, \lambda \leq -1.$$

Hence for $-1 < \lambda < 1$, we get

$$\|\lambda Tx - T\lambda x\| \leq 4|\lambda|^{\frac{1}{2}}\sqrt{\epsilon}\|x\|.$$

Thus for any real number λ, we get

$$\|\lambda Tx - T\lambda x\| \leq 4\sqrt{\epsilon}(|\lambda|^{\frac{1}{2}} + |\lambda|^{\frac{3}{2}})\|x\|.$$

It is easy to see that

$$\begin{aligned}(1-\epsilon)^2\|y\|^2 - \|Tx\|^2 - \|T(x+y)\|^2 \\ \leq -2(Tx, T(x+y)) \\ \leq (1+\epsilon)^2\|y\|^2 - \|Tx\|^2 - \|T(x+y)\|^2,\end{aligned}$$

$$\begin{aligned}(1-\epsilon)^2\|x\|^2 - \|Ty\|^2 - \|T(x+y)\|^2 \\ \leq -2(Ty, T(x+y)) \\ \leq (1+\epsilon)^2\|x\|^2 - \|Ty\|^2 - \|T(x+y)\|^2\end{aligned}$$

and

$$\begin{aligned}(1-\epsilon)^2\|x-y\|^2 - \|Tx\|^2 - \|Ty\|^2 \\ \leq -2(Tx, Ty) \\ \leq (1+\epsilon)^2\|x-y\|^2 - \|Tx\|^2 - \|Ty\|^2.\end{aligned}$$

So we have

$$\|Tx + Ty - T(x+y)\| \leq 4\sqrt{\epsilon}(\|x\| + \|y\|).$$

STEP 4. There is an isometry I from X onto Y for which $\|Ix - Tx\| \leq C(\epsilon)(\|x\|^{\frac{1}{2}} + \|x\|^{\frac{3}{2}})$ where $C(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof. For $x \in X$, there are $\alpha_1, ..., \alpha_n$ such that $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$. We define $I : X \longrightarrow Y$ by $Ix = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$. Then I is an isometry. By Step 3, we have

(1) $\|T(\alpha_1 e_1 + \cdots + \alpha_n e_n) - T(\alpha_1 e_1) - \cdots - T(\alpha_n e_n)\| \leq 8n\sqrt{\epsilon}\|x\|.$

By Step 2, we get
(2)
$$\|Te_i - f_i\| \le \left\|Te_i - \frac{Te_i}{\|Te_i\|}\right\| + \left\|\frac{Te_i}{\|Te_i\|} - f_i\right\| \le \epsilon + C_i(\epsilon) \text{ for } i = 1, 2, ..., n.$$

By (1), (2) and Step 3, we obtain

$$\|Tx - Ix\| \le 8n\sqrt{\epsilon}\|x\| + 4n\sqrt{\epsilon}\left(\|x\|^{\frac{1}{2}} + \|x\|^{\frac{3}{2}}\right) + (n\epsilon + C_1(\epsilon) + \cdots + C_n(\epsilon))n\|x\|.$$

Thus we have
$$\|Tx - Ix\| \le C(\epsilon)(\|x\|^{\frac{1}{2}} + \|x\|^{\frac{3}{2}})$$
where $C(\epsilon) = 12n\sqrt{\epsilon} + n^2\epsilon + n(C_1(\epsilon) + \cdots + C_n(\epsilon))$. This completes the proof of Step 4.

References

1. I. Aharoni and L. Lindenstrauss, *Uniform equivalence between Banach spaces*, Bull. Amer. Math. Soc. **34** (1978), 281-283.
2. K. Jarosz, *Nonlinear generalizations of the Banach-Stone theorem*, Studia Math. **313** (1989), 97-107.
3. K. Jarosz, *Ultraproducts and small bound perturbations*, Pacific Jour. Math. **148** (1991), 81-88.
4. P. Mankiewicz, *On Lipschitz mappings between Fréchet space*, Studia Math. **41** (1972), 225-241.
5. S. Mazur and S. Ulam, *Sur les transformations isométriques d'espaces vectoriels normés*, C. R. Acad. Sci. Paris Sér **194** (1932), 946-948.

Kil-Woung Jun
Department of Mathematics
Chungnam National University
Taejon 305-764, Korea

Dal-Won Park
Department of Mathematics Education
Kongju National University
Kongju 314-701, Korea