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CURVATURE HOMOGENEITY FOR FOUR–DIMENSIONAL MANIFOLDS

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1. Introduction and preliminaries

Let (M, g) be an n-dimensional, connected Riemannian manifold with Levi Civita connection ∇ and Riemannian curvature tensor R defined by

$$R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

for all smooth vector fields X, Y. $\nabla R, \cdots, \nabla^k R, \cdots$ denote the successive covariant derivatives and we assume $\nabla^0 R = R$.

In [17] I.M. Singer studied infinitesimally homogeneous spaces and introduced the following condition :

$P(\ell)$: for every $x, y \in M$ there exists a linear isometry $\phi : T_xM \mapsto T_yM$ such that

$$\phi^*((\nabla^k R)_y) = (\nabla^k R)_x \quad \text{for} \quad k = 0, 1, \cdots, \ell.$$

A Riemannian manifold such that $P(0)$ holds is said to be *curvature homogeneous* and if $P(\ell)$ holds, the manifold is said to be *curvature homogeneous up to order* ℓ. Further, for any point $x \in M$, let G_s^x be the Lie group

$$G_s^x = \{a \in O(T_xM) | (\nabla_x^i R)a = (\nabla^i R)_x, i = 0, 1, ..., s\}.$$

Its Lie algebra $\mathfrak{g}_s^x$ consists of all skew-symmetric endomorphisms A of T_xM such that $A \cdot (\nabla^i R)_x = 0$ for $i = 0, .., s$. Here A acts as a derivation of the tensor algebra. Clearly, there always exists a first

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integer k_x such that $\mathfrak{g}_{k_x} = \mathfrak{g}_{k_x+1}$. Moreover, if $P(\ell)$ is satisfied, then $\mathfrak{g}_i^x$ and $\mathfrak{g}_i^y$ are conjugated for $0 \leq i \leq \ell$. Hence, if $P(k_x + 1)$ holds, k_x does not depend on x. In this case we put $\mathfrak{g}_i^x = \mathfrak{g}_i, k_M = k_x$. A Riemannian manifold satisfying the condition $P(k_M + 1)$ is said to be *infinitesimally homogeneous* [17] and Singer's main result in [17] is the following

THEOREM 1. *A connected, simply connected, complete, infinitesimally homogeneous Riemannian manifold is a homogeneous Riemannian space.*

It is clear that $k_M + 1 \leq \frac{1}{2}n(n-1)$ but a better estimate, namely $k_M + 1 < \frac{3}{2}n$, is given in [3, p. 165].

The following useful lemma also follows from [17].

LEMMA 2. *If $P(r)$ is satisfied, then there exists a maximal principal subbundle F_r^b of the orthonormal frame bundle $O(M,g) \to M$ on which the components $R_{ijk\ell}$ and $R_{h_s \cdots h_1, ijk\ell}, 1 \leq h_1, \cdots, h_s, i, j, k, \ell \leq n, 1 \leq s \leq r$, are constants and which contains a given frame $b \in O(M,g)$. Moreover, the connected component of the identity of $G_r^x, x \in M$ being arbitrary, is the structure group of F_r^b.*

Here we used the notational convection

$$R_{ijk\ell} = g(R_{e_i e_j} e_k, e_\ell),$$

$$R_{h_s \cdots h_1, ijk\ell} = g((\nabla^s_{h_s \cdots h_1} R)_{e_i e_j} e_k, e_\ell)$$

where $\{e_i, i = 1, ..., n\}$ is an orthonormal frame.

There is no lack of examples of non-homogeneous curvature homogeneous manifolds (i.e., (M,g) satisfying $P(0)$). We refer to [6] - [13], [16], [18] - [21] for more details, more references and up-to-date information, in particular for the three- and four-dimensional case.

For dim $M = 3$, Singer's estimate is $k_M + 1 \leq 3$ but in [14] the first author proved the following sharper result :

THEOREM 3. *Let (M,g) be a three-dimensional, connected, simply connected, complete Riemannian manifold which is curvature homogeneous up to order 1. Then (M,g) is homogeneous and moreover, (M,g) is either symmetric or a group space with a left invariant metric.*

We also refer to [5] for a short proof of the homogeneity.

When dim $M = 4$, Singer's estimate gives $k_M + 1 \leq 6$ and Gromov's estimate is $k_M + 1 < 6$. Moreover, we proved in [15] :

THEOREM 4. *Let (M, g) be a four-dimensional, connected simply connected and complete Riemannian manifold which is curvature homogeneous up to order two. Then (M, g) is homogeneous and moreover, (M, g) is either symmetric or a group space with a left invariant metric.*

The second part of this statement is proved in [1], [4].

The main purpose of this note is to prove the following improvement of Theorem 4:

THEOREM 5. *Let (M, g) be a four-dimensional, connected, simply connected and complete Riemannian manifold which is curvature homogeneous up to order one. Then (M, g) is homogeneous and moreover, (M, g) is either symmetric or a group space with a left invariant metric.*

2. Sketch of the proof of the main result

Because of [1], [4] we have only to prove the homogeneity. So, let $u = (e_1, ..., e_n)$ be a smooth local cross section of $O(M, g)$ and put

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \Gamma_{ijk} e_k \qquad , \quad i, j = 1, ..., n.$$

Then the local functions Γ_{ijk} satisfy

$$\Gamma_{ijk} + \Gamma_{ikj} = 0, \qquad i, j, k = 1, ..., n.$$

For dim $M = n = 4$ and $x \in (M, g)$, we may choose an orthonormal basis $\{e_i, i = 1, ..., 4\}$ of $T_x M$ such that

$$Qe_i = \lambda_i e_i, \quad 1 \leq i \leq 4,$$

where Q denotes the Ricci endomorphism and when (M, g) satisfies $P(0)$, all the eigenvalues λ_i are constant on M. Then we have to consider the following five cases :

(I) four different Ricci eigenvalues ,

(II) three different Ricci eigenvalues ,
(III) two Ricci eigenvalues with multiplicity two ,
(IV) three equal Ricci eigenvalues ,
(V) four equal Ricci eigenvalues

and without loss of generality, we may suppose :

(I) $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$,
(II) $\lambda_1 = \lambda_2, \lambda_3 \neq \lambda_4 \neq \lambda_1 \neq \lambda_3$,
(III) $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$,
(IV) $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$,
(V) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$.

We start by considering the cases (I) and (V).

LEMMA A. *An (M,g) of type (I) is homogeneous.*

Proof. For such an (M,g) we have $\mathfrak{g}_0 = \{0\}$ or equivalently, $k_M = 0$. Then the result follows from Singer's theorem.

LEMMA B. *An (M,g) of type (V) is homogeneous.*

Proof. In this case (M,g) is a curvature homogeneous Einstein space and hence symmetric as follows from a still unpublished result of A. Derdziński [2].

So we are left with the cases (II), (III) and (IV). More specifically we have to consider the following subcases :

(1)
$$\begin{aligned} &(II)_1 : \mathfrak{g}_0 = \{0\}, \\ &(II)_2 : \mathfrak{g}_0 = \mathfrak{so}(2) \oplus \{0\} = \mathfrak{g}_1, \\ &(II)_3 : \mathfrak{g}_0 = \mathfrak{so}(2) \oplus \{0\}, \mathfrak{g}_1 = \{0\}; \end{aligned}$$

(2)
$$\begin{aligned} &(III)_1 : \mathfrak{g}_0 = \{0\}, \\ &(III)_2 : \mathfrak{g}_0 = \mathfrak{so}(2) \oplus \{0\} = \mathfrak{g}_1, \\ &(III)_3 : \mathfrak{g}_0 = \mathfrak{so}(2) \oplus \{0\}, \mathfrak{g}_1 = \{0\}; \\ &(III)_4 : \mathfrak{g}_0 = \mathfrak{so}(2) \oplus \mathfrak{so}(2) = \mathfrak{g}_1, \\ &(III)_5 : \mathfrak{g}_0 = \mathfrak{so}(2) \oplus \mathfrak{so}(2)\ ,\ \mathfrak{g}_1 = \mathfrak{so}(2) \oplus \{0\}, \\ &(III)_6 : \mathfrak{g}_0 = \mathfrak{so}(2) \oplus \mathfrak{so}(2)\ ,\ \mathfrak{g}_1 = \{0\}; \end{aligned}$$

$$(3) \qquad \begin{aligned} &(IV)_1 : \mathfrak{g}_0 = \{0\}, \\ &(IV)_2 : \mathfrak{g}_0 = \mathfrak{so}(2) \oplus \{0\} = \mathfrak{g}_1, \\ &(IV)_3 : \mathfrak{g}_0 = \mathfrak{so}(2) \oplus \{0\}, \mathfrak{g}_1 = \{0\}; \\ &(IV)_4 : \mathfrak{g}_0 = \mathfrak{so}(3) = \mathfrak{g}_1, \\ &(IV)_5 : \mathfrak{g}_0 = \mathfrak{so}(3),\ \mathfrak{g}_1 = \mathfrak{so}(2) \oplus \{0\}, \\ &(IV)_6 : \mathfrak{g}_0 = \mathfrak{so}(3),\ \mathfrak{g}_1 = \{0\}; \end{aligned}$$

First we note that the cases $(III)_2, (III)_3$ cannot occur. Further, we have

LEMMA C. *The theorem holds for the cases* $(II)_1$, $(II)_2$, $(III)_1$, $(III)_4, (IV)_1, (IV)_2, (IV)_4$.

Proof. As is Lemma A the result follows at once from Singer's result.

For the six remaining cases we note that the method of proof is similar to the one used in [15] but the explicit computations are now considerably longer. For that reason we only give a brief sketch of the proofs.

LEMMA D. *The theorem holds for the case* $(II)_3$.

Proof. The hypothesis implies that we may choose a global orthonormal frame field $u = (e_1, e_2, e_3, e_4)$ such that $Qe_i = \lambda_i e_i, 1 \leq i \leq 4$, and such that the functions $R_{abcd}(u), R_{i,abcd}(u), 1 \leq i, a \leq 4$, are constant on M. Then it follows by considering the components of the covariant derivative $\nabla\rho$ of the Ricci tensor ρ of type $(0,2)$ that the functions $\Gamma_{i13}, \Gamma_{i14}, \Gamma_{i23}, \Gamma_{i24}, \Gamma_{i34}, 1 \leq i \leq 4$, are also constant. Moreover the frame field may be chosen such that, up to sign, the non-zero components of R are given by

$$(4) \quad \begin{cases} R_{1212} = \alpha, R_{1313} = R_{2323} = \beta, R_{1414} = R_{2424} = \gamma, R_{3434} = \delta, \\ R_{1324} = \epsilon, R_{1423} = -\epsilon, R_{1234} = 2\epsilon. \end{cases}$$

Using (4), the components of ∇R and the Bianchi identities, direct but long computations lead to the consideration of the following cases :

(i) $(\Gamma_{123} + \Gamma_{213}, \Gamma_{124} + \Gamma_{214}, \Gamma_{113} - \Gamma_{223}, \Gamma_{114} - \Gamma_{224}) \neq (0,0,0,0)$;

(ii) $(\Gamma_{123}+\Gamma_{213}, \Gamma_{124}+\Gamma_{214}, \Gamma_{113}-\Gamma_{223}, \Gamma_{114}-\Gamma_{224}) = (0,0,0,0)$ and $(\Gamma_{134}, \Gamma_{234}) \neq (0,0)$;

(iii) $(\Gamma_{123}+\Gamma_{213}, \Gamma_{124}+\Gamma_{214}, \Gamma_{113}-\Gamma_{223}, \Gamma_{114}-\Gamma_{224}) = (0,0,0,0)$ and $(\Gamma_{134}, \Gamma_{234}) = (0,0)$ and $9\epsilon^2-(\alpha-\beta)(\delta-\beta)=0, 9\epsilon^2-(\alpha-\gamma)(\delta-\gamma)\neq 0$ (respectively $9\epsilon^2-(\alpha-\beta)(\delta-\beta)\neq 0, 9\epsilon^2-(\alpha-\gamma)(\delta-\gamma)=0$).

In all these cases one derives that (M,g) is a group space.

LEMMA E. *The theorem holds for the cases* $(III)_5$ *and* $(III)_6$.

Proof. It is already proved in [15] that an (M,g) of type $(III)_5$ is a direct product of two surfaces of different constant curvature. So, in this case, (M,g) is a symmetric space.

Hence, it suffices to consider the case $(III)_6$. Again, the hypothesis implies that we may choose a global orthonormal frame field $u=(e_1,e_2,e_3,e_4)$ such that $Qe_i=\lambda_i e_i, 1\le i\le 4$ and such that the functions $R_{abcd}(u), R_{i,abcd}(u), 1\le i,a\le 4$, are constant. Then the functions $\Gamma_{i13}, \Gamma_{i23}, \Gamma_{i14}, \Gamma_{i24}, 1\le i\le 4$, are constant. Further, the frame field may be chosen such that, up to sign, the non-zero components of R are given by

$$(5)\quad \begin{cases} R_{1212}=\alpha, R_{3434}=\delta, R_{1313}=R_{2323}=R_{1414}=R_{2424}=\beta, \\ R_{1324}=\epsilon, R_{1423}=-\epsilon, R_{1234}=2\epsilon. \end{cases}$$

Now, we proceed as in Lemma D and consider first the case $(\alpha-\beta,\epsilon)\neq(0,0), (\delta-\beta,\epsilon)\neq(0,0)$. Then we get

$$\Gamma_{123}=\Gamma_{213}, \Gamma_{124}=\Gamma_{214}, \Gamma_{324}=\Gamma_{423}, \Gamma_{213}=\Gamma_{314},$$

and further direct computations lead here to the consideration of the following cases :

(i) $(\Gamma_{123}, \Gamma_{113}) \neq \pm(\Gamma_{114}, -\Gamma_{124})$ and $(\Gamma_{314}, \Gamma_{313}) \neq \pm(-\Gamma_{323}, \Gamma_{324})$;

(ii) $(\Gamma_{134}, \Gamma_{313}) = (-\Gamma_{323}, \Gamma_{324})$ respectively $(\Gamma_{323}, -\Gamma_{324})$, and $(\Gamma_{123}, \Gamma_{113}) \neq \pm(\Gamma_{114}, -\Gamma_{124})$;

(ii)' $(\Gamma_{314}, \Gamma_{313}) \neq \pm(-\Gamma_{323}, \Gamma_{324})$ and $(\Gamma_{123}, \Gamma_{113}) = (-\Gamma_{114}, \Gamma_{124})$ respectively $(\Gamma_{114}, -\Gamma_{124})$;

(iii) $(\Gamma_{314}, \Gamma_{313}) = (-\Gamma_{323}, \Gamma_{324})$ respectively $(\Gamma_{323}, -\Gamma_{324})$ and $(\Gamma_{123}, \Gamma_{113}) = (-\Gamma_{114}, \Gamma_{124})$ respectively $(\Gamma_{114}, -\Gamma_{142})$ together with $(\Gamma_{313}, \Gamma_{323}) \neq (0,0)$;
(iv) the same conditions as in (iii) but now writh $(\Gamma_{313}, \Gamma_{323}) = (0,0)$.

In all these cases we find that (M, g) is a group space or a direct product of two surfaces of different constant curvature.

For the case $(\alpha - \beta, \epsilon) \neq (0,0), (\delta - \beta, \epsilon) = (0,0)$ respectively $(\alpha - \beta, \epsilon) = (0,0), (\delta - \beta, \epsilon) \neq (0,0)$, we also deduce the same result.

LEMMA F. *The theorem holds for the cases* $(IV)_3, (IV)_5, (IV)_6$.

Proof. For $(IV)_5$ and $(IV)_6$ the result follows from [15]. In these cases (M, g) is a direct product of a three-dimensional space of non-zero constant curvature and $\mathbb{R}$, and hence it is a symmetric space.

So, we are left with the case $(IV)_3$. Now we choose again a global orthonormal frame field $u = (e_1, e_2, e_3, e_4)$ on M such that $Qe_i = \lambda_i e_i, 1 \leq i \leq 4$, and such that $R_{abcd}(u), R_{i,abcd}(u), 1 \leq i, a \leq 4$ are constant. Then it follows that $\Gamma_{i14}, \Gamma_{i24}, \Gamma_{i34}, 1 \leq i \leq 4$, are constant functions. This frame field may be specialized further such that, up to sign, the non-vanishing components of R are given by

$$(6) \quad \begin{cases} R_{1212} = \alpha, R_{1313} = R_{2323} = \beta, R_{3434} = \delta, R_{1414} = R_{2424} = \gamma, \\ R_{1324} = \epsilon, R_{1423} = -\epsilon, R_{1234} = 2\epsilon. \end{cases}$$

First, let $\epsilon \neq 0$. Then also $\Gamma_{i13}, \Gamma_{i23}, 1 \leq i \leq 4$, are constants. Further direct computations then lead to the following cases :

(i) $(\Gamma_{123} + \Gamma_{213}, \Gamma_{124} + \Gamma_{214}, \Gamma_{113} - \Gamma_{223}, \Gamma_{114} - \Gamma_{224}) \neq (0,0,0,0)$;
(ii) $(\Gamma_{123} + \Gamma_{213}, \Gamma_{124} + \Gamma_{214}, \Gamma_{113} - \Gamma_{223}, \Gamma_{114} - \Gamma_{224}) = (0,0,0,0)$ and $(\Gamma_{134}, \Gamma_{234}) \neq (0,0)$.

In both cases (M, g) turns out to be a group space.

Next, let $\epsilon = 0$. It follows that $\Gamma_{i13}, \Gamma_{i23}, 1 \leq i \leq 4$, are constants. Here we find that (M, g) is a group space or a direct product of a three-dimensional space of non-zero curvature and $\mathbb{R}$.

The main result follows nowdirectly from these lemmas.

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