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ON THE HARRIS ERGODICITY OF A CLASS OF MARKOV PROCESSES

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1. Introduction

Suppose $\{X_n\}$ is a Markov process taking values in some arbitrary state space $(S, \mathcal{F})$ with temporarily homogeneous transition probabilities $p^n(x, A) = P(X_n \in A | X_0 = x)$, $x \in S$, $A \in \mathcal{F}$. Write $p(x, A)$ for $p^1(x, A)$.

As usual, we require the function $x \to p(x, A)$ to be $\mathcal{F}$-measurable for every $A \in \mathcal{F}$.

We call a Markov process with n-step transition probability $p^n(x, A)$ *φ-irreducible* for some nontrivial σ-finite measure φ if whenever $\varphi(A) > 0$,

$$\sum_{n=1}^{\infty} 2^{-n} p^n(x, A) > 0 \text{ for every } x \in S.$$

A probability measure π is said to be *invariant* for p, or for the Markov process X_n, if

$$\pi(A) = \int_S \pi(dy) p(y, A), \quad A \in \mathcal{F} \tag{1}$$

It is important to know whether a Markov process is *ergodic*, i.e., whether there exists a unique invariant probability measure π.

In this paper we are interested in asymptotics of irreducible Markov processes generated by iterations of i.i.d. random maps.

It may be noted that every Markov process on a Borel subset S of a polish space (i.e., a complete separable metric space) may be represented by iterations of i.i.d. random maps on S into S (Kifer [1986]).

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In this article the particular iterations are of the form

$$X_{n+1} = f(X_n) + \varepsilon_{n+1}, \quad n \geq 0 \tag{2}$$

where $\{\varepsilon_n\}$ are i.i.d. random variables on $\mathbb{R}^1$, f an $\mathbb{R}^1$ valued function on $S = \mathbb{R}^1$, X_0 arbitrary (but independent of $\{\varepsilon_n\}$). Here $\mathcal{F} = \mathcal{B}(\mathbb{R}^1)$ is the class of Borel sets of $\mathbb{R}^1$.

Petruccelli and Woolford [1984] have proved that if f is the function defined by

$$f(x) = \alpha x \mathbf{1}_{(x<0)} + \beta x \mathbf{1}_{(x \geq 0)}, \ \varepsilon_1 \text{ has a density}$$

which is positive everywhere and $E\varepsilon_1 = 0$, then $\alpha < 1$, $\beta < 1$ and $\alpha\beta < 1$ are both necessary and sufficient for the existence of a unique invariant probability measure π.

In section 2, we derive a criterion for the ergodicity for general f.

2. Sufficient conditions for the ergodicity of a class of Markov processes

Let $\{X_n\}$ be the Markov process in (2). We say the $p(x, \cdot)$ has the *strong Feller property* if for every $B \in \mathcal{B}(\mathbb{R}^1)$, $p(x, B)$ is a continuous function in x, and $p(x, \cdot)$ has the (*weak*) *Feller property* if for every sequence x_n in $\mathbb{R}^1$ converging to x, $p(x_n, \cdot)$ converges weakly to $p(x, \cdot)$ as $n \to \infty$. It is of interest to establish conditions under which (2) is Harris ergodic. For the general theory, we refer to Tweedie [1975], [1983a].

Our main result is

THEOREM 2.1. *For $\{X_n\}$ in (2), suppose f is a continuous function on $\mathbb{R}^1$, $\{\varepsilon_n\}$ are i.i.d. having a density function $g_1(\cdot)$ on $\mathbb{R}^1$ which is positive everywhere and $E\varepsilon_1 = 0$. Write $\overline{\alpha} \equiv \overline{\lim}_{x \to -\infty} \frac{f(x)}{x}$, $\underline{\alpha} \equiv \underline{\lim}_{x \to -\infty} \frac{f(x)}{x}$, $\overline{\beta} \equiv \overline{\lim}_{x \to \infty} \frac{f(x)}{x}$, $\underline{\beta} \equiv \underline{\lim}_{x \to \infty} \frac{f(x)}{x}$.*

Then each of the followings is a sufficient condition for the existence of a unique invariant probability measure π for $\{X_n\}$:

(i) $\overline{\alpha} < 1, \overline{\beta} < 1, \underline{\beta} > 0, \underline{\alpha} \geq 0;$

(ii) $\overline{\alpha} < 0, \overline{\beta} < 1, \underline{\beta} \geq 0, \underline{\alpha} > -\infty;$

(iii) $\overline{\alpha} < 1, \overline{\beta} < 0, \underline{\beta} > -\infty, \underline{\alpha} \geq 0;$

(iv) $\overline{\alpha} < 0, \overline{\beta} < 0, \underline{\alpha} \cdot \underline{\beta} < 1.$

Before proving Theorem 2.1, we state a corollary which is an immediate consequence of it.

COROLLARY 2.2. *If both limits* $\alpha = \lim_{x \to -\infty} \frac{f(x)}{x}$, $\beta = \lim_{x \to \infty} \frac{f(x)}{x}$, *exist then* $\alpha < 1$, $\beta < 1$, $\alpha\beta < 1$ *is a sufficient condition for the existence of a unique invariant probability for* $\{X_n\}$.

First let us state a proposition whose proof is straightforward and, therefore, omitted.

PROPOSITION 2.3. *For* $\{X_n\}$ *in* (2),
(a) $p(x, \cdot)$ *has the Feller property if* f *is continuous.* (b) $p(x, \cdot)$ *has the strong Feller property if*

(i) *the distribution* Q *of* ε_1 *is absolutely continuous with respect to the Lebesgue measure with a density and*
(ii) f *is continuous.*

We state another proposition proved by Tweedie [1975], [1983a] as follows:

PROPOSITION 2.4. *Let the state space* S *be a metric space with* $\mathcal{F} = \mathcal{B}(S)$*-Borel* σ *field. Assume* $p(x, \cdot)$ *has the Feller property and* p *is* φ*-irreducible with respect to a nontrivial* σ*-finite measure* φ*. Then the Markov process with transition probability* $p(x, \cdot)$ *is Harris ergodic if there exist a nonnegative measurable function* g*, and a compact set* K*, and a constant* $c > 0$ *such that*

$$\int g(y)p(x, dy) \leq g(x) - c \qquad \forall x \in K^c,$$

$$\sup_{x \in K} \int g(y)p(x, dy) < \infty.$$

Proof of Theorem 2.1. For $x \in \mathbb{R}^1$, $A \in \mathcal{B}(\mathbb{R}^1)$,

$$p(x, A) = P(f(x) + \varepsilon_1 \in A) = \int_A g_1(t - f(x))\, dt.$$

Since f is continuous, $p(x, \cdot)$ has the Feller property.

For any given pair of numbers α, β such that $\alpha < 1$, $\beta < 1$, $\alpha\beta < 1$, it is easy to see that there exist $a > 0$, $b > 0$ such that $1 > \beta > -(ab^{-1})$, $1 > \alpha > -(ba^{-1})$.

$$g(x) = \begin{cases} ax & \text{if } x > 0 \\ b|x| & \text{if } x \leq 0. \end{cases}$$

First let $x > 0$. Then

$$\begin{aligned}\int p(x,dy)g(y) =& E\{g(X_{n+1}) \mid X_n = x\} \\ =& a\int_{\{y+f(x)>0\}} yg_1(y)\,dy + af(x)\int_{\{y+f(x)>0\}} g_1(y)\,dy \\ & - b\int_{\{y+f(x)\leq 0\}} yg_1(y)\,dy - bf(x)\int_{\{y+f(x)\leq 0\}} g_1(y)\,dy\end{aligned}$$

Choose θ, $0 < \theta < 1$ such that $\overline{\beta} + \theta < 1$. Since $\overline{\beta} \equiv \overline{\lim}_{x\to\infty} \frac{f(x)}{x}$, $\underline{\beta} \equiv \underline{\lim}_{x\to\infty} \frac{f(x)}{x}$, there exists M_θ such that

$$(\underline{\beta} - \theta)x < f(x) < (\overline{\beta} + \theta)x \qquad \forall x > M_\theta$$

If $\underline{\beta} \geq 0$, then

$$\int p(x,dy)g(y) \leq C + a\left(\overline{\beta} + \left(1 + \frac{b}{a}\right)\theta\right)x \quad \text{for some } C > 0.$$

Choose $0 < \theta' < \theta$ such that $\overline{\beta} + (1 + \frac{b}{a})\theta' < 1$. Since $\overline{\beta} + (1 + \frac{b}{a})\theta' < 1$, we can choose $M_1 > M_\theta$ such that

$$\underline{\beta} > -\infty \text{ and } \int p(x,dy)g(y) \leq g(x) - 1 \qquad \forall x > M_1.$$

If $\overline{\beta} < 0$, $\underline{\beta} > -\infty$ and $\underline{\beta} > -(ab^{-1})$, there exists $\theta_1 > 0$ such that $-b\underline{\beta} < a - \theta_1$ and then we can choose θ, $0 < \theta < 1$, such that

$$\overline{\beta} + \theta < 0, \quad \theta < \frac{\theta_1}{b},$$

so that

$$\int p(x, dy)g(y) \leq C + (a - \theta_1)x \int_{\{y+f(x)\leq 0\}} g_1(y)\,dy$$
$$+ (b\theta - \theta_1)x \int_{\{y+f(x)\leq 0\}} g_1(y)\,dy \quad \forall x > M_\theta.$$

Since $\int_{\{y+f(x)\leq 0\}} g_1(y)\,dy \nearrow 1$ as $x \to \infty$, there exists $M_2 > M_1$ such that $x > M_2$ implies

$$\int p(x, dy)g(y) \leq g(x) - 1.$$

Next let $x \leq 0$. Choose θ, $0 < \theta < 1$, such that $\overline{\alpha} + \theta < 1$. Since $\overline{\lim}_{x\to-\infty} \frac{f(x)}{x} \equiv \overline{\alpha}$, $\underline{\lim}_{x\to-\infty} \frac{f(x)}{x} \equiv \underline{\alpha}$, there exists $M_\theta(> 0)$ such that $x < -M_\theta$ implies $(\overline{\alpha} + \theta)x < f(x) < (\underline{\alpha} - \theta)x$. If $\underline{\alpha} \geq 0$, then

$$\int p(x, dy)g(y) \leq C - bx\left(\overline{\alpha} + \left(1 + \frac{a}{b}\right)\theta\right)$$

for some constant $c > 0$. Choose $0 < \theta' < \theta$ such that

$$\overline{\alpha} + \left(1 + \frac{a}{b}\right)\theta' < 1.$$

By our choice of θ', there exists $M_3 > M_\theta$ such that $x < -M_3$ implies

$$b|x| \cdot \left(\overline{\alpha} + \left(1 + \frac{a}{b}\right)\theta\right) \leq b|x| - C - 1$$

and thus

$$\int p(x, dy)g(y) \leq b|x| - 1$$
$$= g(x) - 1 \quad \text{for} \quad x < -M_3.$$

If $\overline{\alpha} < 0$, $\underline{\alpha} > -\infty$ and $a < \frac{b}{-\underline{\alpha}}$, there exists $\theta_1 > 0$ such that $-a\underline{\alpha} < b - \theta_1$ and then we can choose θ, $0 < \theta < 1$, such that

$$\overline{\alpha} + \theta < 0, \qquad \theta < \frac{\theta_1}{a},$$

so that

$$\begin{aligned}
&\int p(x,dy)g(y)dy \\
&\le C - a\underline{\alpha}|x| \int_{\{y+f(x)>0\}} g_1(y)\,dy \\
&\quad + a\theta|x| \int_{\{y+f(x)>0\}} g_1(y)\,dy \quad \text{for some } C>0 \\
&\le C + (b-\theta_1)|x| \int_{\{y+f(x)>0\}} g_1(y) + a\theta|x| \int_{\{y+f(x)>0\}} g_1(y)\,dy \\
&\le C + b|x| + (a\theta - \theta_1)|x| \int_{\{y+f(x)>0\}} g_1(y)\,dy.
\end{aligned}$$

Since $\int_{\{y+f(x)>0\}} g_1(y)\,dy \nearrow 1$ as $x \to -\infty$, there exists $M_4 > M_3$ such that $x < -M_4$ implies

$$\begin{aligned}
\int p(x,dy)g(y) &\le b|x| - 1 \\
&= g(x) - 1 \quad \text{for} \quad x < -M_4.
\end{aligned}$$

Let $M = \max\{M_2, M_4\}$. Take $K = [-M, M]$. Then

$$\int p(x,dy)g(y) \le g(x) - 1 \quad \text{for} \quad x \in K^c$$

In what follows, we see that the arguments above can fit in each case of (i), (ii),(iii) and(iv) with appropriate constants a, b and M.

Case(i). Since $0 \le \underline{\alpha} \le \overline{\alpha} < 1$, $0 < \underline{\beta} \le \overline{\beta} < 1$ and $\overline{\alpha}\overline{\beta} < 1$, there exist $a_1 > 0, b_1 > 0$ such that $1 > \overline{\beta} > -(a_1 {b_1}^{-1})$, $1 > \overline{\alpha} > -(b_1 {a_1}^{-1})$.

For $x > 0$, the first part of arguments above (with $\underline{\beta} \ge 0$) and for $x \le 0$, the third part (with $\underline{\alpha} \ge 0$) can be directly applied, respectively, to conclude that

$$\int p(x,dy)g(y) \le g(x) - 1 \quad \text{for} \quad x \in K^c,$$

here

$$g(x) = \begin{cases} a_1 x & \text{if } x > 0 \\ b_1|x| & \text{if } x \le 0. \end{cases}$$

Case(ii). Since $-\infty < \underline{\alpha} \le \overline{\alpha} < 0 < 1$, $0 \le \underline{\beta} \le \overline{\beta} < 1$, and $\underline{\alpha}\overline{\beta} \le \overline{\alpha}\overline{\beta} < 1$, there exist $a_2 > 0, b_2 > 0$ such that $1 > \overline{\beta} > -(a_2 {b_2}^{-1})$, $1 > \underline{\alpha} > -(b_2 {a_2}^{-1})$.

For $x > 0$, the first part and for $x \le 0$, the last part of arguments (with $\underline{\alpha} \le \overline{\alpha} < 0$) can be applied, respectively.

Case(iii) By the symmetric argument for $\overline{\alpha}$, $\underline{\beta}$ as case(ii), we can reach the same conclusion.

Case(iv) Since $\underline{\alpha} \le \overline{\alpha} < 0 < 1$, $\underline{\beta} \le \overline{\beta} < 0 < 1$ and $\underline{\alpha}\underline{\beta} < 1$,there exist $a_4 > 0, b_4 > 0$ such that $1 > \underline{\beta} > -(a_4 {b_4}^{-1})$, $1 > \underline{\alpha} > -(b_4 {a_4}^{-1})$.

For $x > 0$, the second part (with $\underline{\beta} \le \overline{\beta} < 0$) and for $x \le 0$, the last part (with $\overline{\alpha} < 0$) of arguments can be applied, respectively.

On the other hand,

$$\begin{aligned}\left|\int g(y)p(x,dy)\right| \le & \int |g(t+f(x))g_1(t)|\,dt \\ \le & a\int_{\{t+f(x)>0\}} (|t| + |f(x)|)|g_1(t)|\,dt \\ & + b\int_{\{t+f(x)\le 0\}} (|t| + |f(x)|)g_1(t)\,dt \\ \le & B < \infty \quad \text{for} \quad x \in K\end{aligned}$$

for some B, since $E|\varepsilon_1| < \infty$ and f is continuous. Since the ε_1 are assumed to have density > 0 everywhere, the process is Lebesgue measure irreducible and aperiodic. Thus, by Proposition 2.4, $\{X_n\}$ is ergodic. Q.E.D.

REMARK. Even when f is not necessarily continuous, under the same hypotheses of Theorem 2.1, $\{X_n\}$ is ergodic if f is compact (i.e., f sends compact sets into relatively compact sets) and $g_1(\cdot)$ is lower semi-continuous. See Tweedie (1983a, Theorem 4, p. 265).

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