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UNIFIED JACKKNIFE ESTIMATION FOR PARAMETER CHANGES IN AN EXPONENTIAL DISTRIBUTION

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1. Introduction

Many authors have utilized an exponential distribution because of its wide applicability in reliability engineering and statistical inferences (see Bain & Engelhart(1987) and Saunders & Mann(1985)). Here we are considering the parametric estmation in an exponential distribution when its scale and location parameters are linear functions of a known exposure level t, which often occurs in the engineering and physical phenomena.

The purpose of this work is to estimate the effects on the scale and location parameters in the exponential distribution when both parameters change a function of environmental dosage, say t. First, we assume an exponential model and estimate the parameters based upon the complete or truncated samples by the maximum likelihood and jackknife methods. The derived estimators will be shown to be asymptotically unbiased and mean square error(MSE)-consistent under a niced condition.

Throughout the numerical evaluations of biases and MSE's of the maximum likelihood estimators and its jackknife estimators for the scale and location parameters in the small sample sizes, the biases and efficiencies of the proposed estimators will be compared each other.

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2. One parameter exponential distribution

We are considering an exponential distribution with the pdf

$$f(x;\sigma(t)) = \frac{1}{\sigma(t)}\exp\left\{-\frac{x}{\sigma(t)}\right\},\ x>0,\ \sigma(t)>0,$$

written by X ~ EXP(σ(t)).

Here we are considering a unified estimation for the parameter change of exposure levels or times in one parameter exponential distribution even when the parameter is a polynomial of t;

$$\sigma(t) = b_0 + b_1\cdot t + \cdots + b_r\cdot t^r,\ t>0,\ b_i>0,\ i=0,1,\cdots,r.$$

2.1 The complete samples

Assume $X_{1j},\cdots,X_{n_j j}$ be a simple random sample(SRS) taken from $X_j \sim$ EXP($\sigma(t_j)$), j = 1, $\cdots$,r+1, and $X_1,\cdots,X_{r+1}$ be independent, $t_i \neq t_k$ for i $\neq$ k.

Define the following notation:

$$\det\ [t_i^0, t_i^1,\cdots,t_i^r] = \begin{vmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^r \\ 1 & t_2 & t_2^2 & \cdots & t_2^r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{r+1} & t_{r+1}^2 & \cdots & t_{r+1}^r \end{vmatrix}.$$

By the maximum likelihood method, we can obtain the maximum likelihood estimators(MLE) for b_j;

$$\hat{b}_j^{(1)} = \frac{\det\left[t_i^0,\cdots,t_i^{j-1},\overline{X}_{.i},t_i^{j+1},\cdots,t_i^r\right]}{det\left[t_i^0,\cdots,t_i^r\right]},\ j=0,1,\cdots,r,$$

where $\overline{X}_{.i} = \frac{1}{n_i}\sum_{k=1}^{n_i} X_{ki}$, i = 1, $\cdots$,r+1.

The expectations and variances of these estimators $\hat{b}_j^{(1)}$ j = 0,$\cdots$,r, are given by

$$E(\hat{b}_j^{(1)}) = b_j \quad \text{and}$$

$$VAR(\hat{b}_j^{(1)}) = \sum_{k=1}^{r+1}\frac{\sigma^2(t_k)\det^2\left[t_i^0,\cdots,t_i^{j-1},t_i^{j+1},\cdots,t_i^r\right]_{i\neq k}}{n_k\det^2[t_i^0,\cdots,t_i^r]},$$

where det $[t_i^0,\cdots,t_i^{j-1},t_i^{j+1},\cdots,t_i^r]_{i\neq k}$ is a minor determinant eliminated k-row and j-column in the determinant, det $[t_i^0,\cdots,t_i^r]$.

Therefore, we get the following.

PROPOSTION 1. *The MLE's $\hat{b}_j^{(1)}$, $j = 0,\cdots,r$, are unbiased and MSE-consistent estimators of b_j, respectively.*

2.2 The truncated samples

For given $t_i \neq t_k$ for $i \neq k$, 1,$\cdots$,r+1, let $X_{1j}, \cdots, X_{k_j j}, \cdots, X_{n_j j}$ be the truncated random sample(TRS) taken from $X_j \sim \text{EXP}(\sigma(t_j))$, and $X_1, \cdots, , X_{r+1}$ be independent, where $X_{1j}, \cdots, X_{k_j j}$ are dead items or item of failures and $X_{k_j+1 j}, \cdots, X_{n_j j}$ are alive items or runouts, j = 1,$\cdots$, r+1, and

$$\sigma(t) = b_0 + b_1 \cdot t + \cdots + b_r \cdot t^r.$$

The likelihood functions are given by

$$L(b_0, \cdots, b_r \mid t_j) = \prod_{i=1}^{k_j} \frac{1}{\sigma(t_j)} \exp\left\{-\frac{X_{ij}}{\sigma(t_j)}\right\} \prod_{i=k_j+1}^{n_j} \exp\left\{-\frac{X_{ij}}{\sigma(t_j)}\right\},$$

and hence, the MLE's $\hat{b}_j^{(2)}$ for b_j, j = 0,$\cdots$,r, are

$$\hat{b}_j^{(2)} = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, n_i\overline{X_{.i}}/k_i, t_i^{j+1}, \cdots, t_i^r\right]}{\det\left[t_i, \cdots, t_i\right]}.$$

If we assume the truncated number K_j - 1 follows a Poisson distribution with mean λ_j and K_j's are independent, j = 1,$\cdots$,r+1, then the expectations and variances of $\hat{b}_j^{(2)}$, j = 0,$\cdots$,r, are given by

$$E(\hat{b}_j^{(2)}) = \frac{\sum_{m=0}^{r} b_m \det\left[t_i^0, \cdots, t_i^{j-1}, (1 - \exp(-\lambda_i))n_i t_i^m/\lambda_i^{j+1}, t_i^{j+1}, \cdots, t_i^r\right]}{\det\left[t_i^0, \cdots, t_i^r\right]} \quad \text{and}$$

$$VAR(\hat{b}_j^{(2)}) = \frac{\sum_{m=1}^{r+1} \det\left[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r\right]_{i \neq m}}{\det\left[t_i^0, \cdots, t_i^r\right]}$$
$$\{n_m(n_m + 1)A(\lambda_m; k_m) - n_m^2(1 - \exp(-\lambda_m))^2/\lambda_m^2\}\sigma^2(t_m),$$

where $A(\lambda_m; k_m) = \sum_{x=0}^{\infty} \lambda_m \exp(-\lambda_m)/((x+1)(x+1)!)$.

Therefore, we get the following.

PROPOSITION 2. *If every truncated number K_j - 1 follows a Poisson distribution with sufficient large mean λ_j and K_j's are independent, $j = 1,\cdots,r+1$, then the MLE's $\hat{b}_j^{(2)}$, $j = 0,\cdots,r$ are asymptotically unbiased and MSE-consistent estimators of b_j, respectively.*

3. Two parameter exponential distribution

Here we are considering two parameter exponential distribution with the pdf

$$f(x;\sigma(t),\mu(t)) = \frac{1}{\sigma(t)} \ \exp\Big(-\frac{x-\mu(t)}{\sigma(t)}\Big), \ x > \mu(t), \ \sigma(t) > 0,$$

written by X $\sim$ EXP($\sigma(t),\mu(t)$).

We are considering a unified jackknife estimation for the parameter change exposure levels or times in two parameter exponential distribution even when two parameters are polynomials of t

$$\begin{aligned} \sigma(t) =& b_0 + b_1 \cdot t + \cdots + b_r \cdot t^r, \\ \mu(t) =& a_0 + a_1 \cdot t + \cdots + a_r \cdot t^r, \end{aligned}$$

where $t > 0, \ a_i > 0, \ b_i > 0, \ i = 0, 1, \cdots, r$.

3.1 The complete samples

3.1.A The maximum likelihood method

Assume $X_{1j},\cdots,X_{n_j j}$ be a SRS taken from $X_j \sim$ EXP$(\sigma(t_j),\mu(t_j))$, j = 1, $\cdots$, r+1, and $X_1,\cdots,X_{r+1}$ be independent, $t_i \neq t_k$ for every $i \neq k$.

By the maximum likelihood method, we can obtain the MLE's of a_j and b_j, j = 0,1,$\cdots$,r;

$$\hat{a}_j^{(3)} = \frac{\det\left[t_i^0,\cdots,t_i^{j-1},X_{(1)i},t_i^{j+1},\cdots,t_i^r\right]}{\det\left[t_i^0,\cdots,t_i^r\right]} \quad \text{and}$$

$$\hat{b}_j^{(3)} = \frac{\det\left[t_i^0,\cdots,t_i^{j-1},\overline{X_{.i}} - X_{(1)i},t_i^{j+1},\cdots,t_i^r\right]}{\det\left[t_i^0,\cdots,t_i^r\right]}$$

where $X_{(1)j}$, j = 1,$\cdots$,r+1, is the smallest order statistic among X_{1j}, $\cdots$, $X_{n_j j}$.

The expectations and variances of these MLE's $\hat{a}_j^{(3)}$ and $\hat{b}_j^{(3)}$, j = 0,1,$\cdots$,r, are given by

$$E[\hat{a}_j^{(3)}] = a_j + \sum_{k=o}^{r} b_k \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, t_i^r/n_i, t_i^{j+1}, \cdots, t_i^r\right]}{\det\left[t_i^0, \cdots, t_i^r\right]},$$

$$VAR[\hat{a}_j^{(3)}] = \sum_{k=1}^{r+1} \sigma^2(t_k) \frac{\det^2\left[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r\right]_{i \neq k}}{n_k^2 \det^2\left[t_i^0, \cdots, t_i^r\right]},$$

$$E[\hat{b}_j^{(3)}] = \sum_{k=0}^{r} b_k \frac{\det\left[t_i, \cdots, t_i^{j-1}, (n_i - 1)t_i^k/n_i, t_i^{j+1} \cdots, t_i^r\right]}{\det\left[t_i^0, \cdots, t_i^r\right]} \quad \text{and}$$

$$VAR[\hat{b}_j^{(3)}] = \sum_{k=1}^{r+1} \sigma^2(t_k) \frac{(n_k - 1)^2 \det^2\left[t_i^0, \cdots t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r\right]_{i \neq k}}{n_k^2 \det^2\left[t_i^0, \cdots, t_i^r\right]}.$$

Therefore, by taking limits for those expressions.

PROPOSITION 3. *The MLE's $\hat{a}_j^{(3)}$ and $\hat{b}_j^{(3)}$, j = 0,1,$\cdots$,r, are asymptotically unbiased and MSE-consistent estimators for a_j and b_j, respectively.*

3.1.B The jackknife method

By definition of the jackknife method, we can obtain the jackknife estimators of $\hat{a}_j^{(3)}$ and $\hat{b}_j^{(3)}$, for every j = 0,1, $\cdots$, r,

$$J(\hat{a}_j^{(3)}) = n.\hat{a}_j^{(3)} - (n. - 1)\overline{\hat{a}_j^{(3)-1}}$$
$$= \frac{\det\ \left[t_i^0, \cdots, t_i^{j-1}, ((2n. - 1)X_{(1)i} - (n. - 1)X_{(2)i}), t_i^{j+1}, \cdots, t_i^r\right]}{\det\ \left[t_i^0, \cdots, t_i^r\right]} \quad \text{and}$$

$$J(\hat{b}_j^{(3)}) = n.\hat{b}_j^{(3)} - (n. - 1)\overline{\hat{b}_j^{(3)-i}}$$
$$= \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, (n.\overline{X}_{.i} - (2n. - 1)X_{(1)i} + (n. - 1)X_{(2)i}), t_i^{j+1}, \cdots, t_i^r\right]}{n.\ \det\ \left[t_i^0, \cdots, t_i^r\right]},$$

where n. $= n_1 + n_2 + \cdots + n_{r+1}$.

Also, the expectations and variances of these jackknife estimators can be obtained by, for every $j = 0, 1, \cdots, r$,

$$E[J(\hat{a}_j^{(3)})] = a_j - \frac{1}{n.\det\left[t_i^0, \cdots, t_i^r\right]} \sum_{k=o}^{r} b_k \det\left[t_i^0, \cdots, t_i^{j-1},\right.$$
$$\left.(n. - n_i)t_i^k / \{n_i(n_i - 1)\}, t_i^{j+1}, \cdots, t_i^r\right],$$

$$VAR[J(\hat{a}_j^{(3)})] = \frac{1}{n.^2\det^2\left[t_i^0, \cdots, t_i^r\right]} \sum_{k=1}^{r+1} \sigma^2(t_k)\{(2n. - 1)(n_k - 1)^2$$
$$+ (n. - 1)^2(2n_k^2 - 2n_k + 1)\} / \{n_k^2(n_k - 1)^2\}$$
$$\det^2\left[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r\right]_{i \neq k},$$

$$E[J(b_j^{(3)})] = \frac{1}{n.\ \det\ \left[t_i^0, \cdots, t_i^r\right]} \cdot \sum_{k=0}^{r} b_k \det\left[t_i^0, \cdots, t_i^{j-1},\right.$$
$$\left.\{n.(n_i^2 - n_i + 1) - n_i\} t_i^k / \{n_i(n_i - 1)\}, t_i^{j+1}, \cdots, t_i^r\right] \quad \text{and}$$

$$VAR[J(b_j^{(3)})] = \frac{1}{n.^2\ \det^2\left[t_i^0, \cdots, t_i^r\right]} \cdot \sum_{k=1}^{r+1} \sigma^2(t_k)\{n.^2 n_k^2 + (n. - 2)n.$$
$$- (2n. + 1)^2 n_k^2 - (n.^2 - 2n. - 2)\} / \{n_k^2(n_k - 1)^2\}$$
$$\det^2\left[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r\right]_{i \neq k}.$$

Therefore, by taking limits,

PROPOSITION 4. *The jackknife estimators $J(\hat{a}_j^{(3)})$ and $J(\hat{b}_j^{(3)})$, $j = 0, \cdots, r$, are asymptotically unbiased and MSE-consistent estimators for a_j and b_j, respectively.*

Biases and mean square errors(MSE) of the ML estimators and their jackknife estimators for a_j's and b_j's in the small complete samples can be numerically evaluated.

3.2 The truncated samples

For given $t_i \neq t_k$ for $i \neq k$, 1,2,$\cdots$,r+1, let $X_{1j}, \cdots, X_{k_j j}, \cdots, X_{n_j j}$ be the the TRS taken from $X_j \sim$ EXP($\sigma(t_j), \mu(t_j)$), and $X_{1j}, \cdots, X_{k_j j}$ are dead items or items of failures and $X_{k_j+1 j}, \cdots, X_{n_j j}$ are alive items or runouts, $j = 1, \cdots, r+1$, and $X_1, \cdots, X_{r+1}$ be independnet.

The likelihood functions are given by

$$L(a, b \mid t_j)$$
$$= \prod_{i=1}^{k_j} \frac{1}{\sigma(t_j)} \exp\left\{-\frac{X_{ij} - \mu(t_j)}{\sigma(t_j)}\right\} \prod_{i=k_j+1}^{n_j} \exp\left\{-\frac{x_{ij} - \mu(t_j)}{\sigma(t_j)}\right\},$$

and hence, the MLE's $\hat{a}_j^{(4)}$ and $\hat{b}_j^{(4)}$ for a_j and b_j, j = 0,$\cdots$,r, are given by

$$\hat{a}_j^{(4)} = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, X_{(1)i}, t_i^{j+1}, \cdots, t_i^r\right]}{\det\left[t_i^0, \cdots, t_i^r\right]} \quad \text{and}$$

$$\hat{b}_j^{(4)} = \frac{\det\left[t_i^0, \cdots, t_i^{j-1}, \left(\overline{X}_{.i} - X_{(1)i}\right)/k_i, t_i^{j+1}, \cdots, t_i^r\right]}{\det\left[t_i^0, \cdots, t_i^r\right]}.$$

If we assume the truncated number K_j -1 follows a Poisson distribution with mean λ_j, $j = 1, \cdots, r+1$ and K_j's are independent, then the expectations and variances of $\hat{a}_j^{(4)}$, $j = 0, 1, \cdots, r$, are the same as those of $\hat{a}_j^{(3)}$, because the $\hat{a}_j^{(3)}$ and $\hat{a}_j^{(4)}$ are equal and the expectations and variances of $\hat{b}_j^{(4)}$ are given by, for $j = 0, 1, \cdots, r$,

$$E(\hat{b}_j^{(4)})$$
$$= \frac{\sum_{k=0}^r b_k \det\left[t_i^0, \cdots, t_i^{j-1}, (n_i - 1)(1 - \exp(-\lambda_i)) t_i^k/\lambda_i, t_i^{j+1}, \cdots, t_i^r\right]}{\det\left[t_i^0, \cdots, t_i^r\right]},$$

$$VAR(\hat{b}_j^{(4)})$$
$$= \frac{1}{\det^2\left[t_i^0, \cdots, t_i^r\right]} \sum_{m=1}^{r+1} \sigma^2(t_m)\{n_m(n_m - 1)A(\lambda_m; k_m)$$
$$- (n_m - 1)^2 (1 - \exp(-\lambda_m))^2/\lambda_m^2\} \det^2\left[t_i^0, \cdots, t_i^{j-1}, t_i^{j+1}, \cdots, t_i^r\right]_{i \neq k},$$

where $A(\lambda_m; k_m) = \sum_{x=0}^{\infty} \lambda_m^x \exp(-\lambda_m)/((x+1)(x+1)!)$.

From the expectations and variances, we get the following.

PROPOSITION 5. *If every truncated number K_j -1 follows a Poisson distribution with sufficient large mean λ_j and K_j's are independent, $j = 1, \cdots, r$, then the MLE's $\hat{a}_j^{(4)}$ and $\hat{b}_j^{(4)}$ of a_j and b_j, $j = o, \cdots, r$, are asymptotically unbiased and MSE-consistent, respectively.*

Biases and MSE's for the truncated ML estimator for the scale parameter in small truncated samples can also numerically evaluated. Throughout the exact numerical evaluations of biases and MSE's, the jackknife technique is very useful in the bias reduction, but MLE's are more efficient in the small samples than truncated ML-estimators and the jackknife estimators.

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