J. Korean Math. Soc. **32** (1995), No. 1, pp. 41–50

CAUCHY PROBLEM FOR THE EULER EQUATIONS OF A NONHOMOGENEOUS IDEAL INCOMPRESSIBLE FLUID II

SHIGEHARU ITOH

1. Introduction

Let us consider the system of equations

$$(1.1)\qquad \begin{cases} \rho_t + v\cdot\nabla\rho = 0, \\ \rho[v_t + (v\cdot\nabla)v] + \nabla p = \rho f, \\ \text{div } v = 0, \end{cases}$$

in $Q_T = R^3 \times [0,T]$, subject to the initial conditions

$$(1.2)\qquad \begin{cases} \rho|_{t=0} = \rho_0(x), \\ v|_{t=0} = v_0(x). \end{cases}$$

Here $f(x,t)$, $\rho_0(x)$ and $v_0(x)$ are given, while the density $\rho(x,t)$, the velocity vector $v(x,t) = (v^1(x,t), v^2(x,t), v^3(x,t))$ and the pressure $p(x,t)$ are unknowns. The system (1.1) describes the motion of a nonhomogeneous ideal incompressible fluid.

The aim of the present paper is to establish the unique solvability, local in time, of the problem (1.1) and (1.2). This will be carried out by applying the method of the Galerkin approximations. Furthermore, we declare that the assumptions to $\rho_0(x)$ are weakened compared with those in the previous paper [2], in which we proved the similar result by showing the existence of a fixed point of some map.

Our theorem is the following.

Received September 8, 1993.

THEOREM 1.1. *Assume that*

$$\inf \rho_0(x) \equiv m > 0 \quad \text{and} \quad \sup \rho_0(x) \equiv M < \infty, \tag{1.3}$$

$$\nabla \rho_0(x) \in H^2(R^3), \tag{1.4}$$

$$v_0(x) \in H^3(R^3) \quad \text{and} \quad \text{div } v_0 = 0 \tag{1.5}$$

and

$$f(x,t) \in L^\infty(0,T; H^3(R^3)) \quad \text{and} \quad \text{div } f \in L^\infty(0,T; L^1(R^3)). \tag{1.6}$$

Then there exists $T^* \in (0,T]$ such that the problem (1.1) and (1.2) has a unique solution(ρ, v, p) which satisfies

$$m \le \rho(x,t) \le M \tag{1.7}$$

and

(1.8)

$$\begin{aligned}(\nabla \rho, v, \nabla p) \in & L^\infty(0,T^*; H^2(R^3)) \\ & \times L^\infty(0,T^*; H^3(R^3)) \times L^\infty(0,T^*; H^3(R^3)).\end{aligned}$$

In section 2, we establish an a priori estimate of solutions, and then theorem will be proved in section 3.

2. A priori estimate

Let (ρ, v, p) be a sufficiently regular solution. Hereafter c is the generic constant related to the imbedding theorems and c_1 is the one dependent only on m, M and c.

LEMMA 2.1. *For $\rho(x,t)$, the estimates*

$$m \le \rho(x,t) \le M \tag{2.1}$$

and

$$\frac{d}{dt} \|\nabla \rho(t)\|_2 \le c \|v(t)\|_3 \|\nabla \rho(t)\|_2 \tag{2.2}$$

hold, where $\|\bullet\|_k = \|\bullet\|_{H^k(R^3)}$.

Proof. By means of the classical method of characteristics, we have the representation

$$\rho(x,t) = \rho_0(\, y(\tau,x,t)|_{\tau=0}), \tag{2.3}$$

where $y(\tau,x,t)$ is the solution of the Cauchy problem

$$\begin{cases} \dfrac{dy}{d\tau} = v(y,\tau), \\ y|_{\tau=t} = x. \end{cases} \tag{2.4}$$

It results from this that the estimate (2.1) holds.

Let α be a multi-index. We apply the operator

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \left(\frac{\partial}{\partial x_3}\right)^{\alpha_3}$$

on each side of the first equation of (1.1). If we multiply the result by $D^\alpha \rho$, integrate over R^3 and sum over $1 \leq |\alpha| \leq 3$, then we have the equality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\rho\|_2^2 = {} & - \sum_{|\alpha|=1}^{3} \Big[\int_{R^3} v\cdot\nabla(D^\alpha\rho)(D^\alpha\rho)dx \\ & + \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \int_{R^3} D^\beta v\cdot\nabla(D^{\alpha-\beta}\rho)(D^\alpha\rho)dx \Big]. \end{aligned} \tag{2.5}$$

The first term of the right hand side is zero, as is seen by an integration by parts using div $v = 0$. The second term can be estimated by using the inequalities

$$\|gh\|_0 \leq c\,\|g\|_2\,\|h\|_0 \tag{2.6}$$

and

$$\|gh\|_0 \leq c\,\|g\|_1\,\|h\|_1\,, \tag{2.7}$$

and then we get

$$\begin{aligned}(2.8)\qquad &\sum_{|\alpha|=1}^{3}\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\left|\int_{R^3} D^{\beta}v\cdot\nabla(D^{\alpha-\beta}\rho)(D^{\alpha}\rho)dx\right| \\ &\leq c\,\|\nabla v\|_2\,\|\nabla\rho\|_2^2\leq c\,\|v\|_3\,\|\nabla\rho\|_2^2\,.\end{aligned}$$

Hence we find that

$$(2.9)\qquad \frac{d}{dt}\,\|\nabla\rho(t)\|_2^2\leq c\,\|v(t)\|_3\,\|\nabla\rho(t)\|_2^2\,.$$

□

LEMMA 2.2. *If we put*

$$(2.10)\qquad \eta(x,t)=\rho(x,t)^{-1},$$

then the estimates

$$(2.11)\qquad M^{-1}\leq\eta(x,t)\leq m^{-1}$$

and

$$(2.12)\qquad \frac{d}{dt}\,\|\nabla\eta(t)\|_2\leq c\,\|v(t)\|_3\,\|\nabla\eta(t)\|_2$$

hold.

Proof. We can easily see that $\eta(x,t)$ satisfies the equation

$$(2.13)\qquad \begin{cases}\eta_t+v\cdot\nabla\eta=0,\\ \eta|_{t=0}=\rho_0(x)^{-1}\equiv\eta_0(x).\end{cases}$$

Therefore the estimates directly follow from Lemma 2.1. □

LEMMA 2.3. *For $p(x,t)$, the estimate*

$$\begin{aligned}(2.14)\qquad &\|\nabla p(t)\|_3\leq c_1\,(1+\|\nabla\rho(t)\|_2+\|\nabla\eta(t)\|_2)^3 \\ &\qquad\left(\|v(t)\|_3^2+\|f(t)\|_3+\|\ div\ f(t)\|_{L^1(R^3)}\right)\end{aligned}$$

holds.

Proof. Applying the divergence operator on both sides of the second equation of (1.1), we get

$$\text{div } (\eta \nabla p) = - \sum_{i,j=1}^{3} v^i_{x_j} v^j_{x_i} + \text{ div } f \equiv F. \tag{2.15}$$

If we multiply this equation by p and integrate over R^3, then we obtain
(2.16)

$$\begin{aligned} M^{-1} \|\nabla p\|_0^2 &\le \|F\|_{L^{6/5}(R^3)} \|p\|_{L^6(R^3)} \le c \|F\|_{L^{6/5}(R^3)} \|\nabla p\|_0 \\ &\le c \|F\|_{L^1(R^3)}^{2/3} \|F\|_0^{1/3} \|\nabla p\|_0 \le c(\frac{2}{3} \|F\|_{L^1(R^3)} + \frac{1}{3} \|F\|_0) \|\nabla p\|_0 . \end{aligned}$$

Hence we get

$$\|\nabla p\|_0 \le c_1(\|F\|_{L^1(R^3)} + \|F\|_0). \tag{2.17}$$

In order to accomplish our purpose, we use the following inequality (cf. [3]):

$$\|u\|_2 \le \sqrt{\frac{3}{2}}(\|\Delta u\|_0 + \|u\|_0) \quad \text{for any } u \in H^2(R^3). \tag{2.18}$$

Noting that (2.15) can be written in the form $\Delta p = \rho F - \rho \nabla \eta \cdot \nabla p$, we get that for α with $|\alpha| = 2$,

$$\begin{aligned} \|D^\alpha p\|_2 &\le \sqrt{\frac{3}{2}}(\|D^\alpha(\rho F - \rho \nabla \eta \cdot \nabla p)\|_0 + \|D^\alpha p\|_0) \\ &\le \sqrt{\frac{3}{2}}(\|D^\alpha(\rho F)\|_0 + \|D^\alpha(\rho \nabla \eta \cdot \nabla p)\|_0 + \|D^\alpha p\|_0). \end{aligned} \tag{2.19}$$

By the direct calculation, we obtain

$$\|D^\alpha(\rho F)\|_0 \le c(M + \|\nabla \rho\|_2) \|F\|_2 \tag{2.20}$$

and

$$\|D^\alpha(\rho \nabla \eta \cdot \nabla p)\|_0 \le c(M + \|\nabla \rho\|_2) \|\nabla \eta\|_2 \|\nabla p\|_2 , \tag{2.21}$$

and then we get

$$\|D^\alpha p\|_2 \le c_1[(1+\|\nabla\rho(t)\|_2+\|\nabla\eta(t)\|_2)\|F\|_2 + (1+\|\nabla\rho(t)\|_2+\|\nabla\eta(t)\|_2)^2\|\nabla p\|_2]. \tag{2.22}$$

Now, from the interpolation inequality and Young's inequality, we have

$$c_1(1+\|\nabla\rho(t)\|_2+\|\nabla\eta(t)\|_2)^2\|\nabla p\|_2 \le \frac{1}{2}\|\nabla p\|_3 + c_1(1+\|\nabla\rho(t)\|_2+\|\nabla\eta(t)\|_2)^3\|\nabla p\|_0. \tag{2.23}$$

Therefore we find that

$$\|\nabla p\|_3 \le c_1(1+\|\nabla\rho(t)\|_2+\|\nabla\eta(t)\|_2)^3(\|F\|_{L^1(R^3)}+\|F\|_2). \tag{2.24}$$

On the other hand, it is easy to verify that

$$\|F\|_2+\|F\|_{L^1(R^3)} \le c(\|\nabla v\|_2^2+\|\operatorname{div} f\|_2+\|\operatorname{div} f\|_{L^1(R^3)}). \tag{2.25}$$

Consequently, the desired estimate is established. □

LEMMA 2.4. *For $v(x,t)$, the estimate*

$$\frac{d}{dt}\|v(t)\|_3 \le c_1[\|v(t)\|_3^2+(1+\|\nabla\eta(t)\|_2)\|\nabla p(t)\|_3+\|f(t)\|_3] \tag{2.26}$$

holds.

Proof. We rewrite the second equation of (1.1) in the form $v_t+(v\cdot\nabla)v+\eta\nabla p=f$. Applying the operator D^α on each side of this equation, multiplying the result by $D^\alpha v$, integrating over R^3 and summing over $0\le|\alpha|\le 3$, then we have the equality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_3^2 = &-\sum_{|\alpha|=0}^{3}\Big[\int_{R^3}(v\cdot\nabla D^\alpha v)\cdot D^\alpha v\,dx \\ &+\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_{R^3}(D^\beta v\cdot\nabla D^{\alpha-\beta}v)\cdot D^\alpha v\,dx \\ &+\int_{R^3}\eta D^\alpha\nabla p\cdot D^\alpha v\,dx \\ &+\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_{R^3}D^\beta\eta D^{\alpha-\beta}\nabla p\cdot D^\alpha v\,dx\Big] \\ &+\int_{R^3}D^\alpha f\cdot D^\alpha v\,dx.\end{aligned} \tag{2.27}$$

Similarly to the proof of Lemma 2.1, we obtain

(2.28) $$\int_{R^3} (v \cdot \nabla D^\alpha v) \cdot D^\alpha v dx = 0,$$

(2.29)
$$\sum_{|\alpha|=0}^{3} \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \left| \int_{R^3} (D^\beta v \cdot \nabla D^{\alpha-\beta} v) \cdot D^\alpha v dx \right| \leq c \|\nabla v\|_2^3 \leq c \|v\|_3^3 ,$$

(2.30) $$\sum_{|\alpha|=0}^{3} \left| \int_{R^3} \eta D^\alpha \nabla p \cdot D^\alpha v dx \right| \leq m^{-1} \|\nabla p\|_3 \|v\|_3 ,$$

(2.31) $$\sum_{|\alpha|=0}^{3} \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \left| \int_{R^3} D^\beta \eta D^{\alpha-\beta} \nabla p \cdot D^\alpha v dx \right|$$
$$\leq c \|\nabla \eta\|_2 \|\nabla p\|_2 \|\nabla v\|_2 \leq c \|\nabla \eta\|_2 \|\nabla p\|_3 \|v\|_3$$

and

(2.32) $$\sum_{|\alpha|=0}^{3} \left| \int_{R^3} D^\alpha f \cdot D^\alpha v dx \right| \leq \|f\|_3 \|v\|_3 .$$

Hence we find that the estimate (2.26) holds. □

LEMMA 2.5. *There exist* $T^* \in (0, T]$ *such that*

(2.33) $$\sup_{0\leq t\leq T^*} (\|\nabla \rho(t)\|_2 + \|\nabla \eta(t)\|_2 + \|v(t)\|_3 + \|\nabla p(t)\|_3) \leq C.$$

Here T^* *and* C *depend only on* m, M, $\|f\|_{L^\infty(0,T;H^3(R^3))}$, $\|\operatorname{div} f\|_{L^\infty(0,T;L^1(R^3))}$, $\|\nabla \rho_0\|_2$, $\|v_0\|_3$ *and the constants of imbeddings.*

Proof. If we set

(2.34) $$Y(t) = 1 + \|\nabla \rho(t)\|_2 + \|\nabla \eta(t)\|_2 + \|v(t)\|_3$$

and

(2.35) $$K = 1 + \|f\|_{L^\infty(0,T;H^3(R^3))} + \|\operatorname{div} f\|_{L^\infty(0,T;L^1(R^3))},$$

then, from the above lemmas, we have a differential inequality

(2.36) $$\frac{dY(t)}{dt} \le c_1 K Y(t)^6.$$

Therefore we conclude that

(2.37)
$$Y(t) \le Y(0)(1 - 5c_1 K Y(0)^5 t)^{-1/5} \quad \text{provided} \quad t < (5c_1 K Y(0)^5)^{-1},$$

and thus

(2.38) $$Y(t) \le 2Y(0) \quad \text{for} \quad t \le T^* \equiv \frac{31}{160 c_1 K Y(0)^5}.$$

□

3. Proof of theorem 1.1

Since the proof of uniqueness is standard, we will just show the existence of a solution. Let $H^3_\sigma(R^3)$ be the closure of $J(R^3) = \{u \in \{C_0^\infty(R^3)\}^3;\ \operatorname{div} u = 0\}$ in $H^3(R^3)$. Since $H^3_\sigma(R^3)$ is separable and $J(R^3)$ is dense in $H^3_\sigma(R^3)$, there exists $\{\phi^j(x)\} \subset J$, which is total in $H^3_\sigma(R^3)$. We may assume that $\phi^j(x)$ are orthogonal in $H^3(R^3)$.

We apply the Galerkin method with this $\{\phi^j(x)\}$. Namely, we look for $\rho^N(x,t)$, $v^N(x,t) = \sum_{j=1}^{N} a_j^N(t)\phi^j(x)$ and $p^N(x,t)$ satisfying

(3.1)
$$\begin{cases} \rho_t^N + v^N \cdot \nabla \rho^N = 0, \\ ((v_t^N + (v^N \cdot \nabla) v^N + \dfrac{1}{\rho^N} \nabla p^N, \phi^j)) = ((f, \phi^j)), \quad j = 1, \cdots, N, \\ \operatorname{div}(\frac{1}{\rho^N} \nabla p^N) = - \sum_{i,j=1}^{3} (v^N)^i_{x_j} (v^N)^j_{x_i} + \operatorname{div} f, \\ \rho^N\big|_{t=0} = \rho_0(x), \\ v^N\big|_{t=0} = P_N v_0(x), \end{cases}$$

where $((\bullet, \bullet))$ stands for the scalar product in $H^3(R^3)$ and P_N is the orthogonal projection in H^3 on the space spanned by $\phi^1, \cdots, \phi^N$.

The second equations of (3.1) form a system of ordinary differential equations for $a_j^N(t)$ and the fifth one gives the initial conditions.

If we multiply the second equations of (3.1) by $a_j^N(t)$ and add in $j = 1, \cdots, N$, then we obtain the relation

$$((v_t^N + (v^N \cdot \nabla)v^N + \eta^N \nabla p^N, v^N)) = ((f, v^N)), \tag{3.2}$$

where $\eta^N = (\rho^N)^{-1}$. This coincides with (2.27) replacing v, p and η by v^N, p^N and η^N respectively. Therefore, from the results in section 2, we have

$$m \leq \rho^N(x, t) \leq M \tag{3.3}$$

and

$$\sup_{0 \leq t \leq T^*} (\|\nabla \rho^N(t)\|_2 + \|\nabla \eta^N(t)\|_2 + \|v^N(t)\|_3 + \|\nabla p^N(t)\|_3) \leq C. \tag{3.4}$$

Moreover, since ϕ^j are orthogonal in $H^3(R^3)$, we deduce from the second equations of (3.1) that

$$v_t^N = P_N(f - (v^N \cdot \nabla)v^N - \eta^N \nabla p^N). \tag{3.5}$$

Hence we get

(3.6)

$$\|v_t^N(t)\|_2 \leq c_1[\|f(t)\|_2 + \|v^N(t)\|_3^2 + (1 + \|\nabla \eta^N(t)\|_2)\|\nabla p^N(t)\|_2],$$

and with (3.4) it is easily found that

$$\sup_{0 \leq t \leq T^*} \|v_t^N(t)\|_2 \leq C. \tag{3.7}$$

These estimates guarantee the unique solvability of the problem (3.1) on the interval $[0, T^*]$, and furthermore permit to pass to the limit in the nonlinear terms using a standard compactness theorem (cf. [1], [4], [5]). Hence we can verify the existence of a unique solution of the problem (1.1) and (1.2) on the interval $[0, T^*]$ as well as the applicability of the inequalities (2.1) and (2.33) to it. This completes the proof.

References

1. S. N. Antontsev, A. V. Kazhikhov and V. N. Monakhov, *Boundary value problems in mechanics of nonhomogeneous fluids*, North-Holland, Amsterdam-New York-Oxford-Tokyo, 1990.
2. S. Itoh, *Cauchy problem for the Euler equations of a nonhomogeneous ideal incompressible fluid*, J. Korean Math. Soc. **31** (1994), 367-373.
3. O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, 1969.
4. J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non lináires*, Dunod-Gauthier-Villars, Paris, 1969.
5. R. Temam, *Navier-Stokes equations*, North-Holland, Amsterdam-NewYork-Oxford, 1984.

Department of Mathematics
Hirosaki University
Hirosaki 036, Japan
e-mail: sitoh@@fed.hirosaki-u.ac.jp