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FINITENESS PROPERTIES OF SOME POINCARÉ DUALITY GROUPS

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A space Y is called *finitely dominated* if there is a finite complex K such that Y is a retract of K in the homotopy category, i.e., we require maps $i : Y \to K$ and $r : K \to Y$ with $r \circ i \simeq \mathrm{id}_Y$. The following questions are very classical in topology.

QUESTION 1. Does a finitely dominated $K(G,1)$ space have the homotopy type of a finite complex?

QUESTION 2. Suppose that we have a torsion-free, finite extension $1 \to G' \to G \to Q \to 1$ of groups, and suppose that G' acts freely and simplicially on a contractible simplicial complex X' with quotient a finite complex. Can one extend the G'-action on X' to a free and simplicial G-action on X', or find a contractible free G-simplicial complex X so that the quotient X/G has the homotopy type of a finite complex?

We can change the above topological questions to algebraic ones. Let k denote a commutative ring with unit $1 \neq 0$. A group G is called *of type FP over* k or k *is a* kG*-module of type FP* if the trivial G-module k admits a *finite projective resolution* over kG, i.e., there is an exact sequence of kG-modules

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow k \longrightarrow 0$$

with each P_i finitely generated projective. A group G is *of type FL over* k if k admits a *finite free resolution* over kG. Clearly the groups

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of type FL are groups of type FP. Some examples of groups of type FP over $\mathbb{Z}$ or FL over $\mathbb{Z}$ can be obtained using topology, cf. [2]:

(a) If there exists a $K(G,1)$ which is a finite complex, then G is of type FL over $\mathbb{Z}$.

(b) If there exists a finitely dominated $K(G,1)$, then G is of type FP over $\mathbb{Z}$.

(c) If G is a finitely presented group, then the converses of (a) and (b) are true.

When we take the algebraic point of view Question 1 is following: There are no known examples of groups of type FP over $\mathbb{Z}$ which are not of type FL over $\mathbb{Z}$. Algebraically Question 2 means that if G' is of type FL over $\mathbb{Z}$ it is not known whether G is of type FL over $\mathbb{Z}$.

In this paper we will be concerned with the above algebraic situation, which is known as Serre's Conjecture ([6, p.85]), for the groups of type FP over $\mathbb{Q}$ or FL over $\mathbb{Q}$. We will show that the Fuchsian groups $\langle x_1, \cdots, x_q \mid x_i^{n_i} = \prod x_j = 1\rangle$, where either $q > 3$ or $q = 3$ and $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \leq 1$, are of type FP over $\mathbb{Q}$, but not of type FL over $\mathbb{Q}$ (see Example 12).

DEFINITION 1. The *cohomological dimension* of G over k, denoted $\mathrm{cd}_k G$, is defined as follows: $\mathrm{cd}_k G \leq n$ if $H^i(G; M) = 0$ for all $i > n$ and all kG-module M. Clearly if G is of type FP over k, then $\mathrm{cd}_k G < \infty$.

LEMMA 2. *Let $\Lambda = kG$ and $\Lambda^{(n)}$ be the free kG-module of rank n. Suppose G has a partial free resolution of the form*

$$\Lambda^{(n)} \xrightarrow{\alpha} \Lambda^{(n+1)} \xrightarrow{\beta} k \longrightarrow 0.$$

Then

$$0 \longrightarrow \Lambda^{(n)} \xrightarrow{\alpha} \Lambda^{(n+1)} \xrightarrow{\beta} k \longrightarrow 0$$

is a resolution for G and hence G is of type FL over k with $\mathrm{cd}_k(G) \leq 1$.

Proof. We first assert that there is a Λ-basis $\{x_0, x_1, \cdots, x_n\}$ of $\Lambda^{(n+1)}$ such that $\beta(x_0) = 1$ and $\beta(x_i) = 0$ for $i = 1, 2, \cdots, n$. Choose any Λ-basis $\{y_0, y_1, \cdots, y_n\}$ of $\Lambda^{(n+1)}$. Let $a_i = \beta(y_i)$ and let $\bar{y}_0, \bar{y}_1, \cdots, \bar{y}_n$ generate $k^{(n+1)}$. Consider $\bar{\beta} : k^{(n+1)} \to k$ defined by $\bar{\beta}(\bar{y}_i) = a_i$ for all $i = 0, 1, \cdots, n$. Then $\bar{\beta}$ is surjective. As k is k-free, $\bar{\beta}$ is split. Hence there exists a matrix M with entries in k of degree $n + 1$

such that $M\bar{y}_0, M\bar{y}_1, \cdots, M\bar{y}_n$ generate $k^{(n+1)}$ and $\bar{\beta}(M\bar{y}_i) = 1$ or 0 according as $i = 0$ or $i \neq 0$. Put $x_i = My_i$ for all $i = 0, 1, \cdots, n$. Then $\{x_0, x_1, \cdots, x_n\}$ is a Λ-basis of $\Lambda^{(n+1)}$ such that $\beta(x_0) = 1$ and $\beta(x_i) = 0$ for $i = 1, 2, \cdots, n$.

Thus it follows that $\Lambda^{(n+1)} = \Lambda(x_0) \oplus \Lambda(x_1) \oplus \cdots \oplus \Lambda(x_n)$. Let $\mathrm{I}_G = \ker\{\beta|_{\Lambda(x_0)} : \Lambda(x_0) \to k\}$. Then $\ker(\beta) = \mathrm{I}_G \oplus \Lambda(x_1) \oplus \Lambda(x_2) \oplus \cdots \oplus \Lambda(x_n) = \mathrm{I}_G \oplus \Lambda^{(n)}$. Since $\mathrm{im}(\alpha) = \ker(\beta)$, the composite $\pi \circ \alpha : \Lambda^{(n)} \xrightarrow{\alpha} \mathrm{im}(\alpha) = \ker(\beta) = \mathrm{I}_G \oplus \Lambda^{(n)} \xrightarrow{\pi} \Lambda^{(n)}$, where π is the projection onto $\Lambda^{(n)}$, is well-defined and surjective. Let $N = \ker(\pi \circ \alpha)$. Then we have a split short exact sequence $0 \to N \to \Lambda^{(n)} \xrightarrow{\pi\circ\alpha} \Lambda^{(n)} \to 0$ as $\Lambda^{(n)}$ is free. Hence $\Lambda^{(n)} \cong \Lambda^{(n)} \oplus N$. By a result of Kaplansky (cf. [4, 5]), $N = 0$; thus $\pi \circ \alpha$ is an isomorphism and α is an injection. This proves our lemma. □

DEFINITION 3. A group G of type FP over k is called a *duality group* of dimension n over k if $H^i(G; M) \cong H_{n-i}(G; D \otimes_k M)$ for all $i \in \mathbb{Z}$ and kG-module M. Here, $D = H^n(G; kG)$ as an kG-module and G acts *diagonally* on $D \otimes_k M$: $g \cdot (d \otimes m) = g \cdot d \otimes g \cdot m$. When $D = k$, the group G is called a *Poincaré duality group*. The Poincaré duality group G is called *orientable* if it acts trivially on $D = k$; otherwise, it is called *non-orientable*.

Suppose G is a duality group over k of dimension n. The duality isomorphisms yield that $H^i(G; kG) \cong H_{n-i}(G; D \otimes_k kG) = H_{n-i}(\{1\}; D)$ (Shapiro's lemma, [2, p. 73]) $= 0$ if $i \neq n$ and D if $i = n$ and $\mathrm{cd}_k G \leq n$. In particular, $\mathrm{cd}_k G = n$. Moreover D is a flat k-module. For, if $0 \to L' \to L \to L'' \to 0$ is an exact sequence of k-modules then $0 \to L' \otimes_k kG \to L \otimes_k kG \to L'' \otimes_k kG \to 0$ is exact and gives the exact sequence $H^{n-1}(G; L'' \otimes_k kG) \to H^n(G; L \otimes_k kG) \to H^n(G; L \otimes_k kG)$. Since $H^i(G; L^* \otimes_k kG) \cong H_{n-i}(G; D \otimes_k L^* \otimes_k kG) \cong H_{n-i}(\{1\}; D \otimes_k L^*)$, it follows that $0 \to D \otimes_k L' \to D \otimes_k L$ is exact.

DEFINITION 4. Let P be a finitely generated projective kG-module. Then the *Hattori-Stallings rank* γ_P of P over k is defined and has a finite expression ([2, pp. 231–241])

$$\gamma_P = \sum_{\tau \in C} \gamma_P(\tau) \cdot [\tau],$$

where $\mathcal{C}$ is a set of representatives for the conjugacy classes in G and $[\tau]$ denotes the conjugacy class of τ. Thus γ_P can be viewed as a function $\gamma_P : \mathcal{C} \to k$ which is constant on each of conjugacy class and is zero for almost all conjugacy classes. If $P \cong kG^{(n)}$, then $\gamma_P = \gamma_P(1)\cdot[1] = n\cdot[1]$.

Suppose G is of type FP over k. Let $0 \to P_n \to \cdots \to P_0 \to k \to 0$ be a finite projective resolution of k over kG. Then the *homological Euler characteristic* $\tilde{\chi}(G)$ of G over k is defined to be

$$\tilde{\chi}(G) = \sum_{i=0}^{n}(-1)^i \sum_{\tau\in\mathcal{C}} \gamma_{P_i}(\tau) \in k.$$

By H. Bass [1], if k is a field then $\tilde{\chi}(G) = \sum_{i=0}^{n}(-1)^i\beta_i(G)$ where $\beta_i(G) = \dim_k H_i(G,k)$. See [1] for more details.

DEFINITION 5. A group G is called *residually finite* if for all $g \neq 1$ in G, there is a normal subgroup K of G with $g \notin K$ such that G/K is finite. A group G is called *Hopfian* if $G \cong G/N$ implies $N = 1$.

It is known that the free groups are residually finite, and finitely generated residually finite groups are Hopfian. Also, fundamental groups of 2-manifolds are residually finite (see [3] for a simple proof). Clearly virtually residually finite groups are themselves residually finite. Hence Fuchsian groups are residually finite. From this, it follows that certain 3-manifolds (e.g., bundles and Seifert fibered spaces) have residually finite fundamental groups.

THEOREM 6. *Let G be an orientable Poincaré duality group of dimension 2 over k, where k is a field of characteristic 0. Then:*

(i) *If G is of type FL over k, then $\beta_1(G) \neq 0$.*
(ii) *If G is torsion-free or residually finite, then G is of type FL over k with $\beta_1(G) \neq 0$.*

Proof. Since G is of type FP with $\mathrm{cd}_k G = 2$, we can take a finite free resolution of the form

$$(*) \qquad 0 \longrightarrow P \xrightarrow{\ \gamma\ } kG^{(n)} \xrightarrow{\ \alpha\ } kG \longrightarrow k \longrightarrow 0.$$

Since $H^i(G;kG) = k$ or 0 according as $i = 2$ or $i \neq 2$ and since $H^2(G;kG) = k$ with trivial G-action, by taking $\mathrm{Hom}_G(-,kG)$ to the resolution $(*)$, we obtain a resolution of k over kG:

$$0 \longrightarrow kG \xrightarrow{\alpha^*} kG^{(n)} \xrightarrow{\gamma^*} P^* =: \mathrm{Hom}_G(P, kG) \longrightarrow k \longrightarrow 0.$$

The homological Euler characteristic $\tilde{\chi}(G) = 1 - n + \sum_{\tau \in \mathcal{C}} \gamma_P(\tau) = 1 - n + \sum_{\tau \in \mathcal{C}} r_{P^*}(\tau) = 2 - \beta_1(G)$. Hence we have $\sum_{\tau \in \mathcal{C}} \gamma_P(\tau) = \sum_{\tau \in \mathcal{C}} r_{P^*}(\tau) = (1+n) - \beta_1(G)$.

Suppose G is of type *FL* over k. We can take $P = kG^{(m)}$ for some m in the resolution $(*)$. Hence $\beta_1(G) = (1+n) - m$, which can not be 0 by Lemma 2.

If G is residually finite, then $\sum_{\tau \in \mathcal{C}} r_{P^*}(\tau) = r_{P^*}(1)$ by Corollary 6.10 of [1]. Now suppose G is torsion-free. If $s \neq 1$ in G and $r_{P^*}(s) \neq 0$, then there is an additive subgroup H of G with $s \in H$ such that H is neither a Poincaré duality group nor free by Theorem 8.1.(e) of [1]. If $(G{:}H)$ is finite, then H is also an orientable Poincaré duality group of dimension 2. If $(G{:}H)$ is infinite, then by Strebel ([7]) $\mathrm{cd}_k H$ is at most 1 and so H must be free. Therefore both cases can not happen. Hence $r_{P^*}(s) = 0$ for all $s \neq 1$ and $\sum_{\tau \in \mathcal{C}} r_{P^*}(\tau) = r_{P^*}(1)$.

Thus if G is torsion-free or residually finite, then $r_{P^*}(1) = (1+n) - \beta_1(G)$ is a non-negative integer. By a theorem of Kaplansky (cf. [4, 5]), $P^* \cong kG^{r_{P^*}(1)}$. By Lemma 2, $\beta_1(G) \neq 0$. □

EXAMPLE 7. Let G denote a Fuchsian triangle group $\Delta(p,q,r) = \langle x, y \mid x^p = y^q = (xy)^r = 1 \rangle$. Then G is residually finite and hence Hopfian. Since $H_1(G,\mathbb{Z}) = G/[G,G] = \langle x, y \mid px = qy = rx + ry = 0 \rangle$ is a finite group and so $\beta_1(G) = 0$, by Theorem 6, G is not of type *FL* over $\mathbb{Q}$.

REMARK 8. Let G be a non-orientable Poincaré duality group of dimension n over k. Then $H^n(G; kG)$ is a nontrivial G-module k^*, which induces the action homomorphism $G \to \mathrm{Aut}(k^*)$. Let G' be the kernel of the homomorphism. Suppose G' has finite index in G. For instance, if $k^* = \mathbb{Z}$ then $\mathrm{Aut}(k^*) \cong \mathbb{Z}_2$ and G' is an index 2 subgroup of G. Now

$$\begin{aligned} H^*(G'; kG') &\cong H^*(G; \mathrm{Hom}_{kG'}(kG, kG')) && \text{([2, Proposition III.6.2])} \\ &\cong H^*(G, kG \otimes_{kG'} kG') && \text{([2, Proposition III.5.9])} \\ &\cong H^*(G; kG). \end{aligned}$$

and G' acts trivially on $H^n(G';kG') = H^n(G;kG) = k$. Hence G' is an orientable Poincaré duality group of dimension n over k.

THEOREM 9. *Let G be a non-orientable Poincaré duality group of dimension 2 over k with a finite image under the nontrivial action homomorphism $G \to \mathrm{Aut}(H^2(G;kG))$. If G is of type FL over k, then $\beta_1(G) \neq 0$.*

Proof. As in the proof of Theorem 6, we take a resolution $(*)$. Let k^* be the nontrivial G-module $H^2(G;kG)$. Then $H_2(G,k) \cong H^0(G,k^*) \cong k^{*^G} = \{0\}$, so $\beta_2(G) = 0$. Let G' be the kernel of the action homomorphism $G \to \mathrm{Aut}(k^*)$; then the index $(G{:}G')$ is finite, say ℓ, and by Remark 8, G' is an orientable Poincaré duality group of dimension 2 over k. Since kG is a free kG'-module of rank ℓ, G' is also of type *FL* so that $\beta_1(G') \neq 0$ by Theorem 6, and the resolution $(*)$ for G is also a resolution of k over kG'. Hence $\chi(G) = 1 - \beta_1(G) = m - n + 1$ and $\chi(G') = 1 - \beta_1(G') + 1 = \ell(m - n + 1)$. It follows that $\beta_1(G) \neq 0$. □

Theorem 6 together with the next two propositions, which are well-known (cf. [2]), yields Example 12.

PROPOSITION 10. *Let $1 \to G' \to G \to Q \to 1$ be an exact sequence of groups, where G has no k-torsion (i.e., for every finite subgroup N of G, the order of N is a unit in k) and Q is finite. Then G is of type FP over k if and only if G' is of type FP over k.*

Proof. Any finitely generated projective kG-module can also be regarded as a projective kG'-module, and as such it is still finitely generated since $(G{:}G') = |Q| < \infty$. This proves the "only if" part.

Conversely, suppose that G' is of type *FP* with $\mathrm{cd}_k G' = n$. Consider a resolution of k over kG

$$(1) \qquad 0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow k \longrightarrow 0.$$

Since G' is of type *FP* with $\mathrm{cd}_k G' = n$, there is a finite projective resolution of k over kG'

$$(2) \qquad 0 \longrightarrow P'_n \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow k \longrightarrow 0.$$

Applying Schanuel's lemma to (1) and (2) yields a kG'-isomorphism

$$P_0 \oplus P'_1 \oplus P_2 \oplus P'_3 \oplus \cdots \cong P'_0 \oplus P_1 \oplus P'_2 \oplus P_3 \oplus \cdots .$$

Since the P_j' are finitely generated, we may assume that the P_j and K are finitely generated as kG'-modules and hence as kG-modules. Now it suffices to show that K is a projective kG-module. Complete (1) arbitrarily to a projective resolution over kG

$$\cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow k \longrightarrow 0.$$

so that $K = \operatorname{im}\{P_n \to P_{n-1}\}$. Put $L = \ker\{P_n \to P_{n-1}\}$. Since G has no k-torsion, by Swan ([8]), $\mathrm{cd}_k G = \mathrm{cd}_k G' = n$ and hence $H^{n+1}(G; L) = 0$. The $(n+1)$-cocycle $\partial_{n+1} \in \mathrm{Hom}_{kG}(P_{n+1}, L)$ is a coboundary, i.e., there is $\phi : P_n \to L$ such that $\phi \circ \partial_{n+1} = \partial_{n+1}$. This implies that $P_n \cong L \oplus K$ and hence K is projective. □

PROPOSITION 11. *Let $1 \to G' \to G \to Q \to 1$ be an exact sequence of groups, where G has no k-torsion and Q is finite. Then G is a (Poincaré) duality group over k if and only if G' is a (Poincaré) duality group over k.*

Proof. By Proposition 10, G is of type FP if and only if G' is of type FP. By Remark 8, $H^*(G; kG) \cong H^*(G'; kG')$. Now the proposition follows from cf. [2, Theorem VIII.10.1]. In deed, suppose G' is a duality group over k of dimension n and consider a finite projective resolution for G: $0 \to P_n \to \cdots \to P_0 \to k \to 0$. Since $H^i(G; kG) = H^i(G'; kG') = 0$ if $i \neq n$, taking $\mathrm{Hom}_G(-, kG)$ yields a resolution $0 \to \bar{P}^0 \to \cdots \to \bar{P}^n \to D \to 0$, where $\bar{P}^i = \mathrm{Hom}_G(P_i, kG)$ is a finitely generated projective kG-module and $D = H^n(G; kG)$. For any G-module M, we have the natural isomorphism $u \otimes m \in \bar{P}^i \otimes_G M \xrightarrow{\cong} (x \mapsto u(x) \cdot m) \in \mathrm{Hom}_G(P_i, M)$ and thus $H^i(G; M) \cong \mathrm{Tor}^G_{n-i}(D, M)$. On the other hand, since D is a flat k-module, $0 \to P_n \otimes_k D \to \cdots \to P_0 \otimes_k D \to D \to 0$ is a flat resolution of D over kG (cf. [2, Proposition III.2.2]) and thus $\mathrm{Tor}^G_{n-i}(D, M) = H_{n-i}((P_* \otimes_k D) \otimes_G M) \cong H_{n-i}(P_* \otimes_G (D \otimes_k M)) = H_{n-i}(G; D \otimes_k M)$. Hence G is a duality group over k of dimension n. Similarly we can show the converse. □

EXAMPLE 12. Let G denote a Fuchsian group $\langle x_1, \cdots, x_q \mid x_i^{n_i} = \prod x_j = 1 \rangle$, where either $q > 3$ or $q = 3$ and $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \leq 1$. Then G contains a surface group as a finite index subgroup that is of type FL over $\mathbb{Q}$. By Propositions 10 and 11, G is of type FP over $\mathbb{Q}$

and a Poincaré duality group of dimension 2 over $\mathbb{Q}$. However, since $\beta_1(G) = 0$ as in Example 7, by Theorem 6, G itself is not of type FL over $\mathbb{Q}$.

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