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REDUCTION FACTOR OF MULTIGRID ITERATIONS FOR ELLIPTIC PROBLEMS

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1. Introduction

Multigrid method has been used widely to solve elliptic problems because of its applicability to many class of problems and fast convergence ([1],[3], [9], [10], [11], [12]). The estimate of convergence rate of multigrid is one of the main objectives of the multigrid analysis([1], [2], [5], [6], [7], [8]). In many problems, the convergence rate depends on the regularity of the solution([5], [6], [8]). In this paper, we present an improved estimate of reduction factor of multigrid iteration based on the proof in[6].

2. Elliptic problems in $\mathbf{R}^2$

Let Ω ba a polygonal domain in $\mathbf{R}^2$ and let

$$\begin{aligned} -Lu &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.1}$$

where L is a uniformly elliptic operator and $f \in L^2(\Omega)$. We further assume the solution u satisfies the elliptic regularity: $u \in H_0^{1+\alpha}(\Omega)$. Let $S_h(\Omega)$ be a finite dimensional subspace of $H_0^1(\Omega)$, say, the space of continuous, piecewise linear functions on some triangulation of Ω. We use standard finite element method to solve (2.1), i.e,

$$A(u,\phi) = (f,\phi), \quad \forall \phi \in S_h(\Omega). \tag{2.2}$$

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Let $M_1 \subset \cdots \subset M_J = S_h(\Omega)$ be a nested sequence of subspaces of $S_h(\Omega)$. Let $(\cdot,\cdot)_k$ be the discrete L^2 inner product on M_k. To define multigrid algorithm we need some operators. For $k = 1, \cdots, J$ define $A_k : M_k \to M_k, P_k : M_J \to M_k, P^0_{k-1} : M_k \to M_{k-1}$ via

$$\begin{aligned}(A_k u, \phi)_k &= A(u,\phi), \quad \forall \phi \in M_k,\\ (A_{k-1}P_{k-1}u, \phi)_{k-1} &= A(u,\phi), \quad \forall \phi \in M_{k-1},\\ (P^0_{k-1}u, \phi)_{k-1} &= (u,\phi)_k, \quad \forall \phi \in M_{k-1}.\end{aligned}$$

and define the multigrid algorithm as usual:

Multigrid Algorithm Set $S_1 = A_1^{-1}$. Assume S_{k-1} has been defined and define $S_k(g)$ as follows:

(1) Set $x^0 = 0$ and $c^0 = 0$.

(2) Define x^l for $l = 1, \cdots, m$ by

$$x^l = x^{l-1} + R_k(g - A_k x^{l-1}). \tag{2.3}$$

where R_k is any symmetric smoother.

(3) Define $x^{m+1} = x^m + c^p$ where $c^i, i = 1, \cdots, p$ is given as

$$c^i = c^{i-1} + S_{k-1}[P^0_{k-1}(g - A_k x^m) - A_{k-1}c^{i-1}]. \tag{2.4}$$

(4) Define x^l for $l = m+2, \cdots, 2m+1$ by

$$x^l = x^{l-1} + R_k(g - A_k x^{l-1}). \tag{2.5}$$

(5) Set $S_k(g) = x^{2m+1}$.

One can also define an algoritm without (2.5). Set $N_1 = A_1^{-1}$. Assume N_{k-1} has been defined and define $N_k(g)$ as follows:

Nonsymmetric Algorithm

(1) Set $x^0 = 0$ and $c^0 = 0$.

(2) Define x^l for $l = 1, \cdots, m$ by

$$x^l = x^{l-1} + R_k(g - A_k x^{l-1}). \tag{2.6}$$

(3) Define c^p for, $i = 1, \cdots, p$ given as

$$c^i = c^{i-1} + N_{k-1}[P^0_{k-1}(g - A_k x^m) - A_{k-1}c^{i-1}]. \tag{2.7}$$

(4) Set $N_k(g) = x^m + c^p$.

Let c satisfy $P^0_{k-1}(g - A_k x^m) - A_{k-1}c = 0$. Then

$$\begin{aligned} c^i - c &= c^{i-1} - c - N_{k-1}A_{k-1}(c^{i-1} - c) \\ &= (I - N_{k-1}A_{k-1})(c^{i-1} - c) \\ c^p - c &= (I - N_{k-1}A_{k-1})^p(c^{i-1} - c) \end{aligned}$$

therefore

$$c^p = [(I - N_{k-1}A_{k-1})^p]A^{-1}_{k-1}P^0_{k-1}A_k(x - x^m).$$

Also

$$\begin{aligned} x - x^m &= x - x^{m-1} - R_k A_k(x - x^{m-1}) \\ &= (I - R_k A_k)(x - x^{m-1}) \\ &= K_k^m x. \end{aligned}$$

Thus, from $P^0_{k-1}A_k = A_{k-1}P_{k-1}$, and $Ax = g$,

$$\begin{aligned} (I - N_k A_k)x &= x - N_k g = x - x^m - c^p \\ &= K_k^m x - [I - (I - N_{k-1}A_{k-1})^p]A^{-1}_{k-1}P^0_{k-1}A_k(x - x^m) \\ &= (I - [I - (I - N_{k-1}A_{k-1})^p P_{k-1}])\, P_{k-1}K_k^m x \\ &= [I - P_{k-1} + (I - N_{k-1}A_{k-1})^p P_{k-1}]K_k^m x. \end{aligned}$$

For the symmetric case, we have similarly,

$$I - S_k A_k = K_k^m[(I - P_{k-1}) + (I - S_{k-1}A_{k-1})^p P_{k-1}]K_k^m. \tag{2.8}$$

Thus

$$\begin{aligned} A((I - S_k A_k)u, v) &= A((I - P_{k-1})K_k^m u, K_k^m v) \\ &\quad + (A(I - S_{k-1}A_{k-1})^p P_{k-1}K_k^m u, K_k^m v). \end{aligned} \tag{2.9}$$

We also find the relation between symmetric multigrid algorithm and nonsymmetric multigrid algorithm:

$$A((I - S_k A_k)u, u) = A((I - N_k A_k)u, (I - N_k A_k)u)).$$

3. Estimates of convergence factor

We shall show that for symmetric algoritm

$$A((I - S_k A_k)u, u) \le \delta_k A(u,u) \tag{3.1}$$

and

$$A((I - N_k A_k)u, (I - N_k A_k)u) \le \delta_k A(u,u) \tag{3.2}$$

for nonsymmetric algorithm. For the proof we need two assumptions. First of all, the following regularity and approximation property:

$$A((I - P_{k-1})u, u) \le C_\alpha^2 \left(\frac{\|A_k u\|_k^2}{\lambda_k} \right)^\alpha A(u,u)^{1-\alpha}, \quad u \in M_k, \tag{3.3}$$

where λ_k is the largest eigenvalue of A_k. This follows from the regularity of the solution of the underlying differential equation and the approximation property of the subspaces M_k.

Next, weed have from the smoothing property of R_k,

$$\frac{\|u\|_k^2}{\lambda_k} \le C_R (R_k u, u)_k, \quad u \in M_k. \tag{3.4}$$

Now the following result is from[6].

THEOREM A. *Assume* (3.3) *and* (3.4). *Then* $S_k(\cdot)$ *defined with* $p = 1$ *satisfies* (3.1) *with*

$$\delta_k = 1 - \varepsilon_k$$

$$\varepsilon_k = \frac{m^\alpha}{m^\alpha + M_\alpha (j+k)^{(1-\alpha)/\alpha}}$$

where $\tilde{M}_\alpha = \left(\frac{1+j}{j}\right)^s \frac{C_R(\alpha C_\alpha^2)^{1/\alpha}}{2}$ *and*

$$s = \begin{cases} \frac{1-\alpha}{\alpha}, & \alpha \ge \frac{1}{2} \\ \left(\frac{1-\alpha}{\alpha}\right)^2, & \alpha < \frac{1}{2}. \end{cases}$$

Set

$$M_\alpha = \left(\frac{1 + \tilde{M}_\alpha}{\tilde{M}_\alpha} \right)^{\frac{1-\alpha}{\alpha}}.$$

In this proof, however, the convergence factor δ_k depends heavily on α and $\delta_k \to 1$ very fast as $\alpha \to 0$. In this paper, we try to improve above result by a more careful analysis. We have under same assumptions, THEOREM 1. *$S_J(\cdot)$ defined with $p=1$ satisfies (3.1) with*

$$\delta_k = 1 - \varepsilon_k$$

$$\varepsilon_k = \frac{m^\alpha}{m^\alpha + M_\alpha (j+k)^{(1-\alpha)}} \tag{3.5}$$

where

$$M_\alpha = S_\alpha^{1-\alpha}(C_\alpha^2)\left(\alpha\frac{C_R}{2}\right)^\alpha \tag{3.6}$$

and S_α, D_α are quantities satisfying

$$S_\alpha = \frac{1}{D_\alpha}\left(\frac{A-1}{A}\right)^{-\alpha}. \tag{3.7}$$

In this result we have several choices for S_α. We shall see some of the examples later.

Proof. From (2.9) and the induction hypothesis,

(3.8)

$$\begin{aligned} &A((I-P_{k-1})u,u) \\ &\le A((I-P_{k-1})K_k^m u, K_k^m u) + \delta_{k-1}A(P_{k-1}K_k^m u, K_k^m u). \\ &= (1-\delta_{k-1})A((I-P_{k-1})K_k^m u, K_k^m u) + \delta_{k-1}A(P_{k-1}K_k^m u, K_k^m u). \end{aligned}$$

As in the proof of Theorem A in [6],

$$\begin{aligned} \le &[(1-\delta_{k-1})C_\alpha^2(1-\alpha)\gamma_k^{-\alpha/(1-\alpha)} + \delta_{k-1}]A(K_k^{2m}u,u) \\ &+ (1-\delta_{k-1})C_\alpha^2 C_R \frac{\alpha}{2m}\gamma_k A((I-K_k^{2m})u,u). \end{aligned}$$

Then (3.1) will follow if we choose γ_k so that

$$(1-\delta_{k-1})C_\alpha^2(1-\alpha)\gamma_k^{-\alpha/(1-\alpha)} + \delta_{k-1} \le \delta_k \tag{3.9}$$

$$\text{and}(1-\delta_{k-1})C_\alpha^2 C_R\frac{\alpha}{2m}\gamma_k \le \delta_k. \tag{3.10}$$

Set γ_k so that

$$(1-\delta_{k-1})C_\alpha^2 C_R\frac{\alpha}{2m}\gamma_k = \delta_{k-1}. \tag{3.11}$$

Since $\delta_k \geq \delta_{k-1}$, (3.10) follows as soon as (3.11) holds. We need to check (3.9) which is equivalent to

$$(3.12) \qquad (1-\delta_{k-1})C_\alpha^2(1-\alpha)\gamma_k^{-\alpha/(1-\alpha)} \leq \delta_k - \delta_{k-1}.$$

Let $D(k) = m^\alpha + M_\alpha(j+k)^{1-\alpha}$ and let

$$(3.13) \qquad 1-\delta_k = \varepsilon_k = \frac{m^\alpha}{m^\alpha + M_\alpha(j+k)^{1-\alpha}} = \frac{m^\alpha}{D(k)}, \qquad \alpha < 1.$$

Then
(3.14)

$$\delta_k - \delta_{k-1} = \varepsilon_{k-1} - \varepsilon_k = \frac{M_\alpha m^\alpha}{D(k)D(k-1)}[(j+k)^{1-\alpha} - (j+k-1)^{1-\alpha}].$$

With $A = j+k$, we see that

$$[A^{1-\alpha} - (A-1)^{1-\alpha}] = A^{-\alpha}\left[1 - \left(1 - \frac{1}{A}\right)^{1-\alpha}\right]$$

$$(3.15) \qquad \geq (1-\alpha)A^{-\alpha}.$$

Since

$$(3.16) \qquad \varepsilon_{k-1} - \varepsilon_k = \frac{(1-\alpha)M_\alpha m^\alpha}{D(k)D(k-1)}(j+k)^{-\alpha},$$

The left side of (3.12) is
(3.17)

$$(1-\delta_{k-1})^{\frac{1}{1-\alpha}}(C_\alpha^2)^{\frac{1}{1-\alpha}}(1-\alpha)\left(\frac{\alpha C_R}{2m}\right)^{\frac{\alpha}{1-\alpha}} \cdot \left[\frac{D(k-1)}{M_\alpha(j+k-1)^{1-\alpha}}\right]^{\frac{\alpha}{1-\alpha}}$$

We want to show (3.17)$\leq$ (3.16) which means

(3.18)

$$\frac{1}{D(k-1)}(C_\alpha^2)^{\frac{1}{1-\alpha}}\left(\frac{\alpha C_R}{2M_\alpha}\right)^{\frac{\alpha}{1-\alpha}}(j+k-1)^{-\alpha} \leq \frac{M_\alpha m^\alpha (j+k)^{-\alpha}}{D(k)D(k-1)}$$

or

$$(3.19) \qquad (C_\alpha^2)^{\frac{1}{1-\alpha}}\left(\frac{\alpha C_R}{2M_\alpha}\right)^{\frac{\alpha}{1-\alpha}}(j+k-1)^{-\alpha} \leq \frac{M_\alpha m^\alpha (j+k)^{-\alpha}}{D(k)}$$

Let $\widetilde{M_\alpha}^{\frac{\alpha}{1-\alpha}} = (C_\alpha^2)^{\frac{1}{1-\alpha}} \left(\frac{\alpha C_R}{2}\right)^{\frac{\alpha}{1-\alpha}}$.

Then it suffices to show

$$\widetilde{M_\alpha}^{\frac{\alpha}{1-\alpha}} \leq M_\alpha^{\frac{1}{1-\alpha}} m^\alpha \left(\frac{A-1}{A}\right)^\alpha / D(k), \tag{3.20}$$

$$\widetilde{M_\alpha}^{\frac{\alpha}{1-\alpha}} [m^\alpha + M_\alpha (j+k-1)^{1-\alpha}] \leq M_\alpha^{\frac{1}{1-\alpha}} m^\alpha \left(\frac{A-1}{A}\right)^\alpha$$

Let $\widetilde{M_\alpha}^{\frac{\alpha}{1-\alpha}} = M_\alpha^{\frac{1}{1-\alpha}} \left(\frac{A-1}{A}\right)^\alpha D_\alpha$. Then

$$D_\alpha [m^\alpha + M_\alpha (j+k-1)^{1-\alpha}] \leq m^\alpha$$

And hence

$$D_\alpha M_\alpha (j+k-1)^{1-\alpha} \leq m^\alpha - D_\alpha m^\alpha = (1-D_\alpha) m^\alpha \tag{3.21}$$

Set

$$M_\alpha^{\frac{1}{1-\alpha}} = S_\alpha (C_\alpha^2)^{\frac{1}{1-\alpha}} \left(\frac{\alpha C_R}{2}\right)^{\frac{\alpha}{1-\alpha}}.$$

Then from

$$\widetilde{M_\alpha}^{\frac{\alpha}{1-\alpha}} = M_\alpha^{\frac{1}{1-\alpha}} \left(\frac{A-1}{A}\right)^\alpha D_\alpha$$

we have $D_\alpha = \frac{1}{S_\alpha} \left(\frac{A-1}{A}\right)^{-\alpha}$. Hence (3.21) is equivalent to

$$\frac{1}{S_\alpha} \left(\frac{A-1}{A}\right)^{-\alpha} S_\alpha^{1-\alpha} C_\alpha^2 \left(\frac{\alpha C_R}{2}\right)^\alpha (A-1)^{1-\alpha} \leq (1-D_\alpha) m^\alpha \tag{3.22}$$

It is equivalent to

$$S_\alpha^{-\alpha} \left(\frac{A-1}{A}\right)^{-\alpha} C_\alpha^2 \left(\frac{\alpha C_R}{2}\right)^\alpha (A-1)^{1-\alpha} \leq (1-D_\alpha) m^\alpha \tag{3.23}$$

which holds if S_α is sufficiently large.

Now we give examples of S_α and D_α for which (3.23) holds. Let S_α be any number such that $S_\alpha^\alpha \to S_0$ as $\alpha \to 0$, where S_0 is some large number greater than zero. Then $D_\alpha \to S_0^{-1}$ and since $C_\alpha \to C_0 = 1$, (3.23) becomes

$$\frac{(A-1)}{S_0} \leq (1 - S_0^{-1}).$$

Which holds if $S_0 \geq A$.

EXAMPLE 1. Let $S_\alpha^\alpha = (1 + 2\alpha \ln A)^{1/\alpha}$. Then $S_\alpha^\alpha \to A^2$ and (3.23) holds for α sufficiently small.

EXAMPLE 2. Let

$$S_\alpha^\alpha = \left(\frac{A-1}{A}\right)^{-\alpha} C_\alpha^2 C \left(\frac{\alpha C_R}{2}\right)^\alpha (A-1)^{1-\alpha}$$

for some large constant C.

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