Bull. Korean Math. Soc. **32** (1995), No. 2, pp. 251–257

EXTENSIONS OF DRINFELD MODULES OF RANK 2 BY THE CARLITZ MODULE

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1. Introduction

In the category of t-modules the Carlitz module C plays the role of $\mathbb{G}_m$ in the category of group schemes. For a finite t-module G which corresponds to a finite group scheme, Taguchi [T] showed that Hom(G,C) is the "right" dual in the category of finite t-modules which corresponds to the Cartier dual of a finite group scheme. In this paper we show that for Drinfeld modules (i.e., t-modules of dimension 1) of rank 2 there is a natural way of defining its dual by using the extension of drinfeld module by the Carlitz module which is in the same vein as defining the dual of an abelian varietiey by its $\mathbb{G}_m$-extensions. Our results suggest that the extensions are the right objects to define the dual of arbitrary t-modules.

2. Duality of finite t-modules

In this section we introduce the work of Taguchi [T]. Unfortunately we have to introduce a series of definitions. First we fix notations:

$A = \mathbb{F}_q[T]$ where q is a power of prime,

$K = \mathbb{F}_q(T)$, $K_\infty =$ the completion of K at ∞

$L =$ the completion of the algebraic closure of K_∞,

$\gamma : A \to L$ a fixed embedding, we often write $\vartheta = \gamma(T)$

$C =$ the Carlitz module.

Received January 14, 1994.
1991 AMS Subject Classification: 11G18.
Key words: Drinfeld modules, Carlitz module, duality.
This paper is supported by KOSEF research Grant 91-08-00-07.

Let B be a subring of A. A B-*module scheme* is a pair (G, Ψ) where G is a commutative group scheme over L and $\Psi : B \to End(G)$ is a ring homomorphism such that Ψ_a induces multiplication $\gamma(a)$ on Lie(G) for each $a \in B$. For an $\mathbb{F}_q$-module G we write $\mathcal{E}_G$ for the set of all homomorphisms of $\mathbb{F}_q$-module schemes over L from G to $\mathbb{G}_a$.

DEFINITION 1. An $\mathbb{F}_q$-module (G, Ψ) is a *finite ϕ-module* if $\mathcal{O}_G$ and $\mathcal{E}_G$ are finite dimensional over L with dim $\mathcal{O}_G = q^{\dim \mathcal{E}_G}$, and $\mathcal{E}_G$ generate $\mathcal{O}_G$ as an L-algebra. A *ϕ-sheaf* over L is a pair $(\mathcal{E}, \phi)$ consisting of a finite dimensional L-space $\mathcal{E}$ and an L-linear map $\mathcal{E}^{(q)} \to \mathcal{E}$ where $\mathcal{E}^{(q)}$ denotes the base extension of $\mathcal{E}$ by the q-th power map on L. The association $G \longmapsto \mathcal{E}_G$ is an antiequivalence between the categories of finite ϕ-modules and ϕ-sheaves.

DEFINITION 2. A *finite* t-*module* over L is an A-module scheme (G, Ψ) such that G is killed by some nonconstant polynomial in $\mathbb{F}_q[T]$ and $(G, \Psi|_{\mathbb{F}_q})$ is a ϕ-module. A t-*sheaf* $(\mathcal{E}, \phi, \psi_t)$ on L is the pair of a ϕ-sheaf $(\mathcal{E}, \phi)$ an endomorphism ψ_t of $(\mathcal{E}, \phi)$ which induces multiplication by $\vartheta = \gamma(T)$ on Coker(ϕ). The association $G \longmapsto \mathcal{E}_G$ is an antiequivalence between the categories of finite t-modules and t-sheaves.

For a finite ϕ-module G we can embed G into $E_G := \mathrm{Symm}_L(\mathcal{E}_G)$ which is induced by the surjection $\mathrm{Symm}(\mathcal{E}_G) \longrightarrow \mathcal{O}_G$.

DEFINITION 3. A *finite* v-*module* (G, Ψ, V) over L is a finite t-module scheme (G, Ψ) over L together with a morphism $V : E_G^{(q)} \to E_G$ of $\mathbb{F}_q$-module schemes such that $\Psi_t = (\vartheta + V \circ F_E)$ where F_E is the Frobenius morphism on E_G. A v-*sheaf* $(\mathcal{E}, \phi, v)$ on L is the pair of a ϕ-sheaf $(\mathcal{E}, \phi)$ over L and an L-linear map $v : \mathcal{E} \to \mathcal{E}^{(q)}$ such that $\psi_t = \vartheta + \phi \circ v$ is a t-sheaf on L. As Usual the association $G \longmapsto \mathcal{E}_G$ is an antiequivalence between the categories of finite v-modules and v-sheaves.

EXAMPLE. Let (E, Ψ) be a Drinfeld module over L of rank r. Assume the action of t is given by

$$\psi_t(X) = \vartheta X + a_1 X^q + \cdots + a_r X^{q^r}.$$

Let $G = Ker(\Psi_a)$ for some $a \in A$. we can furnish G with a v-module structure by defining $v \colon \mathcal{E}_G \to \mathcal{E}_G^{(q)}$ by

$$v(X^{q^i}) = X^{q^{i-1}} \otimes (\vartheta^{q^{i-1}} - \vartheta) + X^{q^i} \otimes a_1^{q^i} + \cdots + X^{q^{r+i-1}} \otimes a_r^{q^i}.$$

PROPOSITION 1. *In our situation (namely $\gamma \colon A \to L$ is injective) a finite t-module commes with a unique v-module structure so that the consept of finite t-module is the same as the concept of finite v-module.*

For an L-module $\mathcal{E}$ we let $\mathcal{E}^* := \mathrm{Hom}_L(\mathcal{E}, L)$. If $(\mathcal{E}, \phi, v)$ is a v-sheaf on L then we get the dual of ϕ and v;

$$\phi^* \colon \mathcal{E}^* \to \mathcal{E}^{(q)}, \quad v^* \colon \mathcal{E}^{*(q)} \to \mathcal{E}^*.$$

DEFINITION. We define the *dual of the* v-*sheaf* to be the v-sheaf $(\mathcal{E}^*, v^*, \phi^*)$. If a finite v-module G corresponds to a v-sheaf $(\mathcal{E}, \phi, v)$ then we define the *dual of the* v-*module* G to be the finite v-module G^* corresponding to the v-sheaf corresponding to $(\mathcal{E}^*, v^*, \phi^*)$.

Taguchi justified that G^* is the right dual by proving the following facts [T].

PROPOSITION 2. *Let G be a finite t-module (which is the same as a v-module by proposition 1). Then,*

(i) *the correspondence $G \longmapsto G^*$is functorial,*
(ii) *G^{**} is canonically isomorphic to G.*

THEOREM 1. *Let C be the Carlitz module and G be a finite t-module.*

(i) *Then the functor which associating to an L-algebra B the group $Hom_{t-mod}(Gx_L B,\ Cx_A B)$ is represented by G^*.*
(ii) *There is nondegenerate A-bilinear pairing*

$$G \times G^* \longrightarrow C.$$

3. Extensions of rank 2 Drinfeld modules by the Carlitz module

First we recall some results of [W].

THEOREM 2. *Let E be a drinfeld module of rank r. Then*

(i) *Ext(E,C) is isomorphic to* $L[\tau]/\mathcal{B}$ *where*

$$\mathcal{B} = \{\alpha\psi_t^E - \psi_t^C\alpha \mid \alpha \in L[\tau]\}$$

and $L[\tau]/\mathcal{B} \cong L^r$ *as additive groups.*

(ii) *The t-action on Ext(E,C)* $= L[\tau]/\mathcal{B}$ *is given by right multiplication by* ψ_t^E *or which is the same as left multiplication by* ψ_t^C.

PROPOSITION 3. *Let* μ *be an isogeny of a Drinfeld module E and let* $G = Ker(\mu)$. *Then we have an exact sequence*

$$o \longrightarrow Hom(G,C) \xrightarrow{\delta_\mu} Ext^1(E,C) \xrightarrow{\mu^*} Ext^1(E,C)$$

and of course Hom(G,C) is identified with G^*.

The proofs of Theorem 2 and Proposition 3 are given in [W].

PROPOSITION 4. *Let* μ *be an isogeny of a Drinfeld module E. Then the kernel of* $\mu^* : Ext^1(E,C) \to Ext^1(E,C)$ *is a finite A-module.*

Proof. Let G = Ker (μ: $E \to E$). In the exact sequence of Proposition 3, Hom(G,C) is a finite A-module. Since δ_ϕ is injective we see that Im(δ_μ) is also a finite A-module which is the same as Ker(μ^*).

THEOREM 3. *Let E be a Drinfeld module and* μ *be an isogeny of E. Then* $Im(\delta_\mu)$ *is contained in the subgroup*

$$\Im = \{f \in Ext^1(E,C) = L[\tau]/\mathcal{B} \mid \text{the constant term of } f \text{ is zero}\}$$

of $Ext^1(E,C)$.

Proof. First suppose $\mu = \psi_a^E$ for some $a \in A$. Then the constant term of μ is a which is nonzero (as we may as well assume). By the formula [W, Proposition 4],

$$f\psi_t^E - \psi_t^C f = \delta_\mu(f)\mu$$

we see that the left hand side of the above equality has no constant term. Since the constant of μ is nonzero we have that the constant of $\delta_\mu(f)$ is zero. That is $\mathrm{Im}(\delta_\mu) \subseteq \mathrm{Im}$.

Now suppose that μ is an arbitrary isogeny of E. Choose $a \in A$ so that Ker $(\mu^*) \subseteq$ Ker (ψ_a^E) i.e., a Ker $(\mu^*) = 0$; such an a exists because Ker (μ^*) is a finite A-module of Ext^1 (E, C). Hence we have

$$\mathrm{Im}\ (\delta_\mu) = \mathrm{Ker}\ (\mu^*) \subseteq \mathrm{Ker}\ (\psi_a^E) \subseteq \Im.$$

As required

PROPOSITION 5. *Let E be a Drinfeld module of rank 2 and let*

$$\psi_t^E = \Delta\tau^2 + g\tau + T\tau^0$$

where Δ *and g are in L. Then* $\Im$ *is a Drinfeld module of rank 2.*

Proof. Let $\alpha \in L[\tau]$ be a constant. Then

$$\frac{\alpha}{\Delta}\psi_t^E - \psi_t^C\frac{\alpha}{\Delta} = \alpha\tau^2 + \frac{g\alpha}{\Delta}\tau + gT\tau^0 - ((\frac{\alpha}{\Delta})^p\tau + gT\tau^0).$$

Hence

$$\alpha\tau^2 = (\frac{\alpha}{\Delta})^p - (\frac{g\alpha}{\Delta})\tau \text{ in } \mathrm{Ext}^1(E, C).$$

Now we compute the t-action (given by Theorem 2);

$$\begin{aligned} t(d\tau) &= (\tau + T\tau^0)(d\tau) \\ &= d^p\tau^2 + dT\tau \\ &= (dT - \frac{gd^p}{\delta} + \frac{d^{p^2}}{\Delta^p})\tau. \end{aligned}$$

Hence on $\Im$, t acts as $\frac{\tau^2}{\Delta^p} - (\frac{g}{\Delta})\tau + T\tau^0$.

If E is a Drinfeld module of rank 2 then we will write $\hat{E}$ for $\Im$. With this notation we get

$$\psi_t^{\hat{E}} = \tau^2/\Delta^p - (g/\Delta^p)\tau + T\tau^0.$$

From now on we only consider Drinfeld modules of rank 2.

PROPOSITION 6. (i) *If E_i (i=1,2) are Drinfeld modules of rank 2 and $\mu\colon E_1 \to E_2$ is an isogeny then we have $\hat{\mu}\colon \hat{E}_2 \to \hat{E}_1$ which is given by right multiplication by μ i.e., $\hat{\mu}(d\tau) = (d\tau)\mu$.*

(ii) *The association $E \longmapsto \hat{E}$ is functorial.*

(iii) *The multiplication by Δ is an isomorphism $E \overset{\cong}{\to} \hat{\hat{E}}$.*

Proof. For (i) simply note that $(d\tau)\mu$ has no constant and the element of $\mathcal{B}$ has no constant either. For (ii) we need to show that $(\mu_2 \circ \mu_1)\hat{} = \hat{\mu}_1 \circ \hat{\mu}_2$ for isogenies μ_1 and μ_2. But this follows from the corresponding properties of extensions. For (iii) we already noted that $\psi_t^{\hat{E}} = \tau^2/\Delta^p - (g/\Delta)\tau + T\tau^0$ when $\psi_t^E = \Delta\tau^2 + g\tau + T\tau^0$. Hence the result.

PROPOSITION 7. *If E is a Drinfeld module of rank 2 then $\hat{E}$ is isomorphic (noncanonically) to E.*

Proof. They have the same j-invarant g^{p+1}/Δ.

THEOREM 4. *Let E_i (i=1,2) be Drinfeld modules of rank 2. Let $\mu\colon E_1 \to E_2$ be an isogeny with kernel G. Then the kernel of $\hat{\mu}\colon \hat{E}_1 \to \hat{E}_2$ is canonically identified with the group $G^* = Hom(G, C)$ of Taguchi [T], i.e.,*

$$0 \to G^* \overset{\delta_\mu}{\to} \hat{E}_2 \overset{\mu}{\to} \hat{E}_1$$

is exact.

Proof. The proof follows from the previous propositions.

COROLLARY. *Let E be a Drinfeld module of rank 2, $a \in A$ and let $G = \mathrm{Ker}(\psi_a^E)$. Then there is a nondegenerate A-bilinear pairing*

$$G \times G \longrightarrow C.$$

Proof. Using the isomorphism of Proposition 7 identify Hom(G,C) with G. Now use Theorem 1, (ii).

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