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STATE CLOSURE SPACES FOR AUTOMATA

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1. Introduction

The theory of automata has considerable influences on development of computer systems, their associated softwares and etc. However, it has long been hampered by the lack of standard notations and the scarcity of basic manipulative tools. To improve these problems, the concepts of successor operator δ and source operator σ have been used as tools by Z.Bavel [3], an algebraic structure by W.M.Holcombe [7] and M.Huzino [9], and the concept of topology by J.Chvalina [5].

It is known that weaker forms of a topological structure can be given on a set, for example, a neighborhood closure, a semi closure and a quasi closure. In [5] the quasi closure is called a closure operation, which is a generalization of Kuratowski's closure operator. In this paper we will use it to define a state closure space and investigate properties (the connectivity, separation and so on) of automata in point of view of closure spaces. We expect to improve the difficulties and to understand more easily theories of automata through relationships between closure spaces and automata. Almost all of Lemmas in this section, which are neccessary to obtain results of relations between closure space and automata, are described without their proofs.

Definition 1.1. If u is a single-valued relation on $\mathcal{P}(X)$ ranging in $\mathcal{P}(X)$ where $\mathcal{P}(X)$ is the power set of a set X, then u is said to be a *closure operation* for X if it satisfies the following : for each R,T $\subset$ X,

(C1) $u(\emptyset) = \emptyset$.
(C2) $R \subset u(R)$.
(C3) $u(R \cup T) = u(R) \cup u(T)$.

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The struct (X,u) is called a *closure space*, u(R) is called the *closure* of R $\subset$ X. A subset R of (X,u) is said to be *closed* if $u(\mathrm{R}) = \mathrm{R}$, and *open* if its complement is closed, i.e., if $u(\mathrm{X}-\mathrm{R}) = \mathrm{X}-\mathrm{R}$. Note that a closure operation u is a Kuratowski's closure operator if $u(\mathrm{R}) = u(u(\mathrm{R}))$ for each R $\subset$ X. In that event the closure operation u is said to be *topological*.

The identity relation d on $\mathcal{P}$(X) and the relation i defined by $i(\emptyset) = \emptyset$ and $i(\mathrm{R}) = \mathrm{X}$ for each non-empty subset R of X are clearly closure operations for a set X. (X,d) and (X,i) are called the *discrete* and the *indiscrete* closure space, respectively. In Example 1.10, (X, u) and (Y,v) are closure spaces but not topological spaces.

LEMMA 1.2. *Let (X,u) be a closure space. Then for each* $A \subset X$,

(1) *If* $A \subset B$, *then* $u(A) \subset u(B)$.
(2) $u(A) \subset u(u(A))$.
(3) $u(X) = X$.

LEMMA 1.3. *The family of all closed (resp. open) subsets of a closure space (X,u) is finitely additive and arbitrary multiplicative (resp. arbitrary additive and finitely multiplicative).*

DEFINITION 1.4. Let R be a subset of a closure space (X,u). The set $\mathrm{int}_u(\mathrm{R}) = \mathrm{X} - u(\mathrm{X}-\mathrm{R})$ [simply, int(R)] is called the *interior* of R. N is called a *neighborhood* (shortly, a nbd) of R if $\mathrm{R} \subset \mathrm{int}_u(\mathrm{N})$.

LEMMA 1.5. *We get easily in (X,u) the following : for each* $R,T \subset X$,

(1) $int_u(X) = X$.
(2) $int_u(R) \subset R$.
(3) $int_u(R \cap T) = int_u(R) \cap int_u(T)$.
(4) *Let int be a relation satisfying (1), (2) and (3). Then* $u = \{R \to u(R)\}$ *is a closure operation for X if* $u(R) = X - int(X - R)$, *and* $int = int_u$.

DEFINITION 1.6. A function f from a closure space (X,u) into a closure space (Y,v) is said to be *continuous at* $x \in$ X if for each R $\subset$ X, x $\in u$(R) implies $f(\mathrm{x}) \in v[f(\mathrm{R})]$, and *continuous* if it is continuous at each point of X, equivalently, if $f[u(\mathrm{R})] \subset v[f(\mathrm{R})]$ for each R $\subset$ X.

LEMMA 1.7. *Let (X,u) and (Y,v) be closure spaces. Then* $f : X \to Y$ *is continuous iff* $u(f^{-1}(T)) \subset f^{-1}(v(T))$ *for each* $T \subset Y$.

Proof. Suppose f is continuous. Let $R = f^{-1}(T)$ for any $T \subset Y$, then $f(u(R)) \subset v(T)$. Thus $u(R) \subset f^{-1}[v(T)]$. Conversely, let $T = f(R)$ for arbitrary $R \subset X$ and $R_1 = f^{-1}(T)$. Then $u(R_1) = u[f^{-1}(T)]$ implies $f[u(R_1)] \subset v(T) = v[f(R_1)] = v[f(R)]$. By Lemma 1.2, $u(R) \subset u(R_1)$ since $R \subset R_1$. Hence $f[u(R)] \subset f[u(R_1)] \subset v[f(R)]$.

LEMMA 1.8. *Let f be a function from a closure space (X,u) to a closure space (Y,v). Then the inverse image of any open (resp. closed) subset of Y is open (resp. closed) in X if f is continuous.*

LEMMA 1.9. *Let (X,u) and (Y,v) be respectively a closure space and a topological space. Then the following are equivalent:*

(1) $f : (X,u) \to (Y,v)$ *is continuous.*
(2) $f^{-1}(R)$ *is closed for each closed subset R of Y.*
(3) $f^{-1}(V)$ *is open for each open subset V of Y.*

EXAMPLE 1.10. Let $X = \{p,q,r\}$, $u(\{p\}) = \{p,q\}$, $u(\{q\}) = \{q,r\}$, $u(\{r\}) = \{r\}$, $u(\emptyset) = \emptyset$, $u(X) = X$ and $Y = \{a,b,c\}$, $v(\{a\}) = \{a\}$, $v(\{b\}) = \{a,b\}$, $v(\{c\}) = \{b,c\}$, $v(\emptyset) = \emptyset$, $v(Y) = Y$. Then u and v are closure operations for X and Y, respectively. Let f be a function from (X,u) to (Y,v) defined by $f(p) = \{c\}$, $f(q) = \{a\}$ and $f(r) = \{a\}$. Because $q \in u(\{p\}) = \{p,q\}$ but $f(q) \notin v[f(\{p\})]$, f is not continuous even though the inverse image of each closed subsets of Y is closed. The converse of Lemma 1.8 is thus not true. Note that v is not topological and the inverse image of the closure $\{b,c\}$ of Y is not a closure of X.

2. State Closure Spaces

An automaton is a triple $A = (Q,\Sigma,\delta)$, where Q is a set (of the internal states), Σ is a nonempty set (of the input symbols), and $\delta : Q \times \Sigma \to Q$ is the (next state) transition function satisfying $\delta(q,mn) = \delta(\delta(q,m),n)$ for each $q \in Q$ and for $m,n \in \Sigma$. For $p,q \in Q$ and $m \in \Sigma$, $\delta(p,m) = q$ is interpreted by the fact that a state machine (system), being in state p, goes to the state q if scanning the input symbol m, and we diagram it as

$$\boxed{p} \xrightarrow{m} \boxed{q} \quad \text{and} \quad \boxed{p}\overset{m}{\circlearrowleft} \quad \text{denotes } \delta(p,m) = p.$$

The set generated by Σ under the concatenation is a free monoid over Σ, that is, the set of all strings of finite length of members of Σ, if including the empty string ϵ such that $\delta(q,\epsilon) = q$ for each $q \in Q$. We will denote the free monoid by Σ^*, $\Sigma^+ = \Sigma^* - \{\epsilon\}$ and $\Sigma^\circ = \Sigma \cup \{\epsilon\}$. The transition function δ can be extended to $Q \times \Sigma^*$ if for each x,y $\in \Sigma^*$ and $q \in Q$, $\delta(q,xy) = \delta(\delta(q,x),y)$ and $\delta(q,\epsilon) = q$.

We define an automaton by a triple $A = (Q,\Sigma,\delta)$ where δ is the extended transition. A triple $B = (T,\Sigma,\delta')$ is called a *subautomaton* of A (denoted by $B \ll A$) if $T \subset Q$ and δ' is the restriction $\delta|_{T\times\Sigma^*}$ (we use δ for the $\delta' = \delta|_{T\times\Sigma^*}$ without confusions). For $R \subset Q$, the set $\delta(R) = \{\delta(r,x) : r \in R,\ x \in \Sigma^*\}$ is called the set of *successors* of R $\subset Q$ and $\sigma(R) = \{q \in Q : \delta(q,x) \in R,\ x \in \Sigma^*\}$ is called the *source* of R. $<R> = (\delta(R),\Sigma,\delta')$ is called the automaton *generated* by R. The symbols defined in the above, A, Q (or Q_A), Σ^+, Σ^*, Σ°, δ (or δ_A), σ (or σ_A), ϵ, $<R>$ and $<\{r\}>$ (or $<r>$) will be used generically without ambiguity and specification.

Let $B = (R,\Sigma,\delta') \ll A = (Q,\Sigma,\delta)$ and $R \subset Q$. The following can be found in [1,2,3,10].

(1) A is *discrete* if for each $q \in Q$, $\delta(q) = \{q\}$.
(2) A is *reflexive* if for each $q \in Q$, there is an $x \in \Sigma^*$ such that $x \neq \epsilon$ and $\delta(q,x) = q$.
(3) $B \ll A$ is called *separated* if $\delta(Q-R) \cap R = \emptyset$.
(4) B is said to be *connected* if B has no separated proper subautomata.
(5) A is *strongly connected* if for any $p,q \in Q$, there is an $x \in \Sigma^*$ such that $\delta(p,x) = q$.

DEFINITION 2.1. For $A = (Q,\Sigma,\delta)$ and $R \subset Q$,

(1) $\kappa(R)$ (or $\kappa_A(R)$) $= \{\delta(q,m) : q \in R,\ m \in \Sigma^\circ\}$ is called the set of *immediate successors* of R.
(2) $\kappa^*(R)$ (or $\kappa^*_A(R)$) $= \{q \in Q : \delta(q,m) \in R,\ m \in \Sigma^\circ\}$ is called the *immediate source* of R.

The set functions, κ and κ^*, are different from functions δ and σ which are known in [4,6]. Intuitively, the concept of κ^* (resp. κ) is based on one of an immediate predecessor (resp. successor). Each of $\kappa(R)$ and $\kappa^*(R)$ is a small part of $\delta(R)$ and $\sigma(R)$, respectively. For p,q

$\in$ Q, q $\in \kappa^*(\{p\})$ (resp. q $\in \kappa(\{p\})$) is interpreted by the fact that a state q is an immediate predecessor (resp. successor) of a state p, equivalently, p can be reached (resp. can reach) from q by one of input symbols including ϵ. Actually, $\kappa(R) = R \cup \{\delta(q,m) : q \in Q, m \in \Sigma\}$ and $\kappa^*(R) = \{q \in Q : \delta(q,m) \in R, m \in \Sigma\}$. κ and κ^* are the small parts of δ and σ, respectivly.

THEOREM 2.2. *Let $A = (Q,\Sigma,\delta)$ and $R,T \subset Q$. Then*

(1) $\kappa(\emptyset) = \emptyset$; $\kappa^*(\emptyset) = \emptyset$.
(2) $R \subset \kappa(R)$; $R \subset \kappa^*(R)$.
(3) $\kappa(R \cup T) = \kappa(R) \cup \kappa(T)$; $\kappa^*(R \cup T) = \kappa^*(R) \cup \kappa^*(T)$.

COROLLARY 2.3. *Let $A = (Q,\Sigma,\delta)$ and $R,T \subset Q$. Then*

(1) *If $R \subset T$, then $\kappa(R) \subset \kappa(T)$; $\kappa^*(R) \subset \kappa^*(T)$.*
(2) $\kappa(R \cap T) \subset \kappa(R) \cap \kappa(T)$; $\kappa^*(R \cap T) \subset \kappa^*(R) \cap \kappa^*(T)$.
(3) $\kappa(R) \subset \kappa(\kappa(T))$; $\kappa^*(R) \subset \kappa^*(\kappa^*(T))$.

Proof. They follow immediately from Lemma 1.3, since κ and κ^* satisfies Axioms C1, C2 and C3. $\subset$'s in this Corollary may not be replaced by =, as shown in Example 2.6, because $\kappa(\{p\} \cap \{r,s\}) = \emptyset$, $\kappa(\{p\}) \cap \kappa(\{r,s\}) = \{r\}$, $\kappa(\kappa(\{p\})) = Q$ and $\kappa(\{p\}) = \{p,q,r\}$.

Let $A = (Q,\Sigma,\delta)$, $x = m_1m_2 \ldots m_i \in \Sigma^*$ and $y = m_1m_2 \ldots m_j \in \Sigma^*$ for $m_i \in \Sigma$. Then we obtain easily that for some $r \in R \subset Q$, $\delta(r,x) = p$ and $\sigma(r,y) = q$ can be expressed by the term of compositions of κ's and κ^*'s, respectively, that is, $\delta(r,x) = \kappa(\kappa(\ldots \kappa(r,m_1), \ldots), m_{i-1}), m_i)$ and $\sigma(r,y) = \kappa^*(\kappa^*(\ldots \kappa^*(r,m_1), \ldots), m_{j-1}), m_j)$. We say that $\kappa(\kappa(\ldots\kappa(r,m_1), \ldots), m_{i-1}), m_i)$ is an *exit-chain* from r (denoted by $\kappa^i(r)$) and $\kappa^*(\kappa^*(\ldots \kappa^*(r,m_1), \ldots), m_{j-1}), m_j)$ is an *enter-chain* into r (denoted by $\kappa^{*j}(r)$). Each of the i and j depends on the length of the strings x and y and is thus not unique. The least element of i's (or j's) is called the *chain index* of r. It is easy to prove that for chain indices i and j, $\delta(R) = \bigcup_{r\in R} \kappa^i(r)$ and $\sigma(R) = \bigcup_{r\in R} \kappa^{*j}(r)$

THEOREM 2.4. *Let $A = (Q,\Sigma,\delta)$ be an automaton. Then*

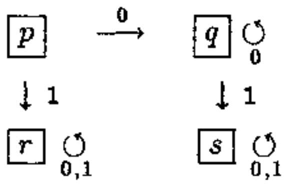
(1) *A is discrete iff $\kappa^1(q) = \kappa^{*1}(q)$ for each $q \in Q$.*
(2) *A is reflexive iff there is a chain index i such that $\kappa^i(q) = \kappa^{*i}(q)$ for each $q \in Q$.*

Proof. (1): A is discrete iff $\delta(q) = \{q\}$ for each $q \in Q$. (2): Let A be reflexive. Then there is a nonempty string $x \in \Sigma^*$ such that $\delta(q,x) = q$ for each $q \in Q$. Suppose $x = m_1m_2$ where $m_1, m_2 \in \Sigma$ without losing generality because we can prove in the same way up to the chain index. In order that $\delta(q,x) = \delta(q,m_1m_2) = \delta(\delta(q,m_1), m_2) = \{q\}$, either (a) $\delta(q,m_1) = q$ and $\delta(q,m_2) = q$, or (b) there is a $p \in Q$ such that $\delta(q,m_1) = p$ and $\sigma(q,m_2) = p$. Hence we have $\kappa^1(q) = \kappa^{*1}(q)$ in the case of (a) and $\kappa^2(q) = \kappa^{*2}(q)$ in the case of (b). The converse is obvious.

Note that κ and κ^* as well as δ and σ satisfy the Axioms C1, C2 and C3. Thus u is a closure operation for Q if defining $u = \{R \to \kappa(R)\}$, $\{R \to \kappa^*(R)\}$, $\{R \to \delta(R)\}$, or $\{R \to \sigma(R)\}$.

DEFINITION 2.5. Let $A = (Q,\Sigma,\delta)$ be an automaton and for $R \subset Q$ let $u = \{R \to \kappa(R)\}$, $\{R \to \kappa^*(R)\}$, $\{R \to \delta(R)\}$, or $\{R \to \sigma(R)\}$. Then the closure space (Q,u) is called a *state closure space associated with* A (simply, a state closure space, or a state closure space with A).

EXAMPLE 2.6. Let $Q = \{p,q,r,s\}$, $\Sigma = \{0,1\}$ and δ be defined by



and $u = \{R \to \kappa^*(R)\}$. Then (Q,u) is a state closure space with $A = (Q,\Sigma,\delta)$. Note that (Q,u) is not a topological space, for $u = \{\emptyset, \{p\}, \{p,q\}, \{p,r\}, \{q,s\}, \{p,q,r\}, \{p,q,s\}, Q\}$.

THEOREM 2.7. *Let $A = (Q,\Sigma,\delta)$ be a automaton and $R \subset Q$. Then $B = (R,\Sigma,\delta) \ll A$ iff $\kappa(R) = R$.*

Proof. Suppose $B \ll A$ and let $\mathcal{O}(R) = \{\delta(r,m)\colon r \in R, m \in \Sigma\}$. Then it is enough to show that $\mathcal{O}(R) \subset R$ since $\kappa(R) = R \cup \mathcal{O}(R)$. Let $q \in \mathcal{O}(R)$. Then there is an $m \in \Sigma$ such that $\delta(r,m) = \kappa(r,m) = q$ for some $r \in R$. Hence $\kappa(r,m) = q \in R$, for $B \ll A$ iff $\delta(R) \in R$ for any $r \in R$ and $x \in \Sigma^*$. Hence $\mathcal{O}(R) \subset R$. Conversely, suppose $B = (R,\Sigma,\delta)$ is not a subautomaton of A. Then there is an exit-chain from some $r \in R$, which has to reach to $q \notin R$, since there is a $q \in \delta(R) - R$. Let the exit-chain be $\kappa^t(R)$. Then there is $m_1, \ldots, m_t \in \Sigma$ such that $\kappa(r,m_1$

$\ldots m_i) \in \delta(R)$ for the $r \in R$. Thus there is an m_{i_0} such that $1 \leq i_0 \leq i$ and $m_{i_0} \in \delta(R) - R$ for the $r \in R$ because the exit-chain $\kappa^i(R)$ has to reach to $q \notin R$. Hence $\kappa(R) \neq R$. It contradicts.

THEOREM 2.8. *Let $A = (Q,\Sigma,\delta)$ and $u^* = \{R \subset Q : \kappa^*(R) = R\}$. Then $\kappa(R) = R$ iff R is open in (Q,u^*).*

Proof. Since u^* satisfies C1, C2 and C3, u^* is a closure operation for Q. Let $\kappa(R) = R$. Then $\kappa(R) \cap (Q-R) = \emptyset$. Thus there is no $m \in \Sigma^\circ$ such that $\kappa(r,m) \in Q-R$ for any $r \in R$, that is, $r \notin \kappa^*(Q-R)$ for each $r \in R$. Since r is arbitrary in R, $\kappa^*(Q-R) \cap R = \emptyset$. This means that $\kappa^*(Q-R) \subset Q - R$. Thus R is open in (Q,u^*). Conversely, let R be open in (Q,u^*). Then $Q - R$ is closed iff $u^*(Q-R) = Q-R$ iff $\kappa^*(Q-R) = Q-R$. Thus $\kappa^*(Q-R) \cap R = \emptyset$. So there is no $m' \in \Sigma^\circ$ such that $\kappa(r,m') \in Q - R$. Thus $\kappa(r) \in R$ for each $r \in R$. We have $\kappa(R) \subset R$.

COROLLARY 2.9. *Let $A = (Q,\Sigma,\delta)$ and $u^* = \{R \subset Q : \kappa^*(R) = R\}$. Then $B = (R,\Sigma,\delta) \ll A$ iff R is open in (Q,u^*).*

COROLLARY 2.10. *Let (Q,u^*) be a state closure space with $A = (Q,\Sigma,\delta)$ where $u^* = \{R \subset Q : \kappa^*(R) = R\}$. Then there is $B = (R,\Sigma,\delta) \ll A$ such that $q \in R \subset N$ iff N is a nbd of $q \in Q$.*

Proof. It is complete if putting $R = \mathrm{int}_{u^*}(R)$ since $\kappa(R) = R$ iff R is open in (Q,u^*).

THEOREM 2.11. *Let $A = (Q,\Sigma,\delta)$ and $u = \{R \subset Q : \kappa(R) = R\}$. Then $\kappa^*(R) = R$ iff R is open in (Q,u).*

Proof. Proofs are similar to those of Theorem 2.8.

COROLLARY 2.12. *Let $u = \{R \subset Q : \kappa(R) = R\}$ and $u^* = \{R \subset Q: \kappa^*(R) = R\}$ for $A = (Q,\Sigma,\delta)$. Then*

(1) *$\kappa(R) = R$ iff R is closed in (Q,u).*
(2) *$\kappa^*(R) = R$ iff R is closed in (Q,u^*).*

COROLLARY 2.13. *Let $B = \{R,\Sigma,\delta\} \ll A = (Q,\Sigma,\delta)$. If $u = \{R \subset Q : \kappa(R) = R\}$ and $u^* = \{R \subset Q : \kappa^*(R) = R\}$, then R is closed (resp. open) in (Q,u) iff R is open (resp. closed) in (Q,u^*).*

THEOREM 2.14. *Let $A = (Q,\Sigma,\delta)$. Then for each $R \subset Q$, $<Q - R> \ll A$ iff R is closed in (Q,u^*).*

Proof. In order to use Theorem 2.6, let $u = \{R \subset Q : \kappa(R) = R\}$ and $u^* = \{R \subset Q : \kappa^*(R) = R\}$. Then $<Q - R> \ll A$ iff $\kappa(Q - R) = Q - R$ by Theorem 2.6 iff $Q - R$ is closed in (Q,u) by Corollary 2.12 iff R is open in (Q,u) iff R is closed in (Q,u^*) by Corollary 2.13 iff $\kappa^*(R) = R$ from Corollary 2.12 iff R is closed in (Q,u^*).

THEOREM 2.15. *Let $A = (Q,\Sigma,\delta)$ and $R \subset Q$. R is separated iff R is open and closed in (Q,u^*).*

Proof. R is separated iff $(R,\Sigma,\delta) \ll A$ and $\delta(Q - R) \cap R = \emptyset$ iff $\kappa(R) = R$ by Theorem 2.7 and $\delta(Q - R) \cap R = \emptyset$ iff R is open in (Q,u^*) by Theorem 2.8 and $\delta(Q - R) \subset Q - R$ iff R is open in (Q,u^*) and $\kappa^*(R) = R$ iff R is open in (Q,u^*) and R is closed in (Q,u^*) by Corollary 2.12

COROLLARY 2.16. *Let $B = (R,\Sigma,\delta) \ll A = (R,\Sigma,\delta)$. Then B is separated iff $\kappa^*(R) = R$.*

THEOREM 2.17. *$A = (Q,\Sigma,\delta)$ is connected iff Q and $\emptyset$ are the only subsets of Q which are both open and closed in (Q,u^*).*

Proof. $A = (Q,\Sigma,\delta)$ is connected iff A has no separated proper subautomata iff there is no proper subset R of Q such that it is open and closed in (Q,u^*) iff Q and $\emptyset$ are the only subsets of Q which are both open and closed in (Q,u^*).

3. Homomorphisms

We define a generalized homomorphism on an automaton.

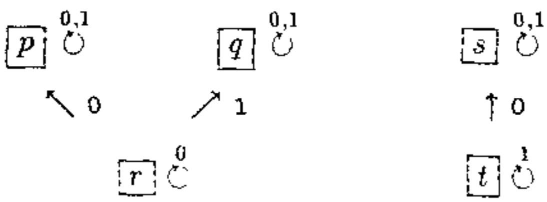
DEFINITION 3.1. Let $A = (Q,\Sigma,\delta_A)$ and $B = (T,\Sigma,\delta_B)$. A function on A to B means a mapping of Q to T and the identity mapping on Σ^*. The function $h : A \to B$ is called a *homomorphism* [3] if it preserves transitions by Σ^*, that is, for any $q \in Q$ and for any $x \in \Sigma^*$, $h(\delta_A(q,x)) = \delta_B(h(q),x)$.

THEOREM 3.2. *Let (Q,u) and (T,v) be state closure spaces with $A = (Q,\Sigma,\delta_A)$ and $B = (T,\Sigma,\delta_B)$, respectively, where $u = \{\kappa(R) \to R\}$ and $v = \{S \subset T : \kappa_B^*(S) = S\}$. Then $f : (Q,u) \to (T,v)$ is continuous if $f : A \to B$ is a homomorphism*

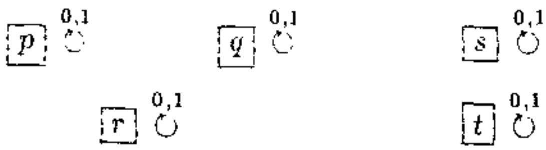
Proof. Let Y be v-closed, then $Y = \kappa_B^*(Y)$. Since $f^{-1}(Y)$ is u-closed iff $u(f^{-1}(Y)) = f^{-1}(Y)$, i.e., $u(f^{-1}(Y)) \subset f^{-1}(Y)$ from Lemma 1.10. Suppose $q \notin f^{-1}(Y)$. It is enough to show $q \notin u(f^{-1}(Y))$. Then $f(q) \notin Y$. Since $Y = \kappa_B^*(Y)$, for any $p \in Y$ and $m \in \Sigma^\circ$, $f(q) \notin \kappa_B^*(p,m)$, thus $f(q) \notin \kappa_B(p,m)$. Since f preserves transitions, it contradicts. Hence $\kappa_B^*(f(q),m) \in Y$ for some $m \in \Sigma$.

EXAMPLE 3.3. Let $Q = \{p,q,r\}$ and $T = \{s,t\}$ be state closure spaces with closures $u = \{Q, \emptyset, \{p\}, \{q\}, \{p,q\}\}$ and $v = \{T, \emptyset, \{s\}\}$, respectively, where $\Sigma = \{0,1\}$. Defining a function $h : Q \to T$ by $h(p)=s$, $h(q)=s$ and $h(r) = t$, then h is continuous. Now let us examine two possible automata.

Case (1)



Case (2)



Then the first h is not a homomorphism although h is continuous, but the second h is a homomorphism. So the converse of Theorem 3.2 may not be true.

CONCLUSION. We have obtained functorial passages between automata and state closure spaces :

(1) A reflexive automaton $A = (Q,\Sigma,\delta)$ corresponds to an indiscrete closure space $(Q,\imath)$.
(2) A discrete automaton $A = (Q,\Sigma,\delta)$ corresponds to the discrete state closure space (Q,d).

(3) A subautomaton corresponds to an open set of (Q,u^*).

(4) Any union of subautomata is a subautomata and the finite intersection of subautomata is also a subautomaton by Lemma 1.3 and by Corollary 2.9.

(5) A connected automaton $A = (Q,\Sigma,\delta)$ in the sense of automaton theory corresponds to the fact that the state closure space (Q,u^*) is connected in the sense of topology by Theorem 2.17.

(6) A a homomorphism between automata $A = (Q,\Sigma,\delta_A)$ and $B = (T,\Sigma,\delta_B)$ under the transitions δ_A and δ_B corresponds to a continuous function from (Q,u) to (T,v).

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