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2–TYPE HYPERSURFACES WITH 1–TYPE GAUSS MAP

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0.Introduction

Submanifolds of finite type were introduced by B. Y. Chen about twelve years ago[1]. These can be regarded as a generalization of both minimal in Euclidean space and minimal into spheres. Finite type submanifolds which are closest in simplicity to minimal ones are the 2-type submanifolds. In [4] it was proved that every 2-type hypersurfaces M of S^{n+1} has non-zero constant mean curvature in S^{n+1} and constant scalar curvature. But relatively little is known about 2-type hypersurfaces in Euclidean space. On the other hand, some authors studied submanifolds with finite type Gauss map. B.Y. Chen and P.Piccini studied compact submanifolds with finite type Gauss map and obtained some results[3]. Recently Y.Kim studied surfaces in 3-dimensional Euclidean space with 1- type Gauss map and he proved that planes, spheres and circular cylinders are the only co-closed surfaces in E^3 with 1-type Gauss map[5]. Through the personal communications with Y.Kim, the author came to know the following geometric problem:

PROBLEM. *What kind of hypersurfaces in Euclidean space with finite type Gauss map are of 2-type?*

In this paper, we shall prove the following theorem, which is a partial solution for the problem.

THEOREM. *A hypersurface M in Euclidean space with 1-type Gauss map is of 2-type if and only if it is a null 2-type hypersurface with constant mean curvature and constant scalar curvature.*

Since the only null 2-type surfaces in Euclidean 3-space are circular cylinders[2], the following corollary is a direct consequence of the theorem.

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COROLLARY. *A surface M in Euclidean 3-space with 1-type Gauss map is of 2-type if and only if it is an open part of a circular cylinder.*

1.Preliminaries

Let M be a hypersurface of a Euclidean space E^{n+1}. It is well known that the position vector field x and the mean curvature vector filed $\overrightarrow{H}$ of M satisfy

$$\Delta x = \overrightarrow{H}, \tag{1.1}$$

where Δ stands for the Laplace operator of the induced metric. The hypersurface M is said to be of k-type if its position vector x can be expressed as

$$x = x_0 + x_1 + x_2 + \cdots + x_k,$$

where x_0 is a constant map, $x_1, \ldots, x_k$ are non-constant maps satisfying $\Delta x_i = \lambda_i x_i$, λ_i being constants, $i = 1, 2, \ldots, k$. In paticular, if $\lambda_1, \ldots, \lambda_k$ are mutually different, we say that M is of k-type. In particular one of $\lambda_1, \ldots, \lambda_k$ is zero, then M is said to be of null k-type. Similarly, a smooth map ϕ of M into E^{n+1} is said to be of finite type if ϕ is a finite sum of E^{n+1}-valued eigenfuntions of Δ(ϕ is not necessarily isometric). It is obvious that for a hypersurface of k-type

$$\Delta^{k-1}\overrightarrow{H} - \sigma_1 \Delta^{k-2}\overrightarrow{H} + \cdots + (-1)^k \sigma_k (x - x_0) = 0, \tag{1.2}$$

where σ_i is the i-th elementary symmetric function of $\lambda_1, \ldots, \lambda_k$. Let ν be the unit normal vector fileld of M. We will view ν as the Gauss map from M to $S^n \subseteq E^{n+1}$. Let H be the mean curvature function of M in E^{n+1} and let A be the Weingarten map associated with ν. Taking into account $\overrightarrow{H} = H\nu$, and computing $\Delta\nu$ and $\Delta\overrightarrow{H}$ one finds

$$\Delta\nu = -gradH - trA^2\nu, \tag{1.3}$$

$$\Delta\overrightarrow{H} = -(2A + H \cdot I)(gradH) + (\Delta H - HtrA^2)\nu, \tag{1.4}$$

where $gradH$ denotes the gradient of H.

2.Proof of theorem

Let M be a 2-type hypersurface with 1-type Gauss map. Then we have the following two equations from (1.2),(1.3),(1.4) and the finite type conditions on M.

(2.1)
$$-(2A+H\cdot I)(gradH)+(\Delta H-HtrA^2)\nu=(\lambda_1+\lambda_2)\overrightarrow{H}-\lambda_1\lambda_2 x,$$

$$-gradH-trA^2\nu=a(\nu-c), \tag{2.2}$$

where λ_1,λ_2,a are constants and c is a constant vector in E^{n+1}. Note that we take $x_0=0$ in equation (2.1). Comparing the tangential and normal components in (2.1) and (2.2) we get

$$\Delta H-HtrA^2=(\lambda_1+\lambda_2)H-\lambda_1\lambda_2\langle x,\nu\rangle, \tag{2.3}$$

$$(2A+H\cdot I)(gradH)=\lambda_1\lambda_2 x^T, \tag{2.4}$$

$$trA^2=a\langle c,\nu\rangle-a, \tag{2.5}$$

$$gradH=ac^T, \tag{2.6}$$

where $(\quad)^T$ means the projection to the tangent space of M and $\langle,\rangle$ is the usual Euclidean metric. From (2.6), it follows that $H=a\langle x,c\rangle+b$ for some constant b. From (1.1),(2.5) and this we get

$$\Delta H=H(trA^2+a). \tag{2.7}$$

This and (2.3) imply

$$(\lambda_1+\lambda_2-a)H=\lambda_1\lambda_2\langle x,\nu\rangle. \tag{2.8}$$

Acting the Laplacian Δ on both sides of (2.8), we have

$$(\lambda_1+\lambda_2-a)\Delta H=\lambda_1\lambda_2\{-a\langle x,c^T\rangle-trA^2\langle x,\nu\rangle-H\}. \tag{2.9}$$

Now suppose that M is of null 2-type, i.e. one of λ_1 and λ_2 is zero. So we may assume that $\lambda_1=0$ and $\lambda_2=\lambda$. Then we have from (2.4) and (2.8)

$$A(gradH)=-\frac{1}{2}HgradH, \tag{2.10}$$

$$\lambda=a. \tag{2.11}$$

From (2.5) and (2.11) we find that $grad(trA^2) = -A(gradH)$. This and (2.10) imply that

$$trA^2 - \frac{1}{4}H^2 = \alpha, \tag{2.12}$$

where α is a constant. Using (2.5), (2.7),(2.11) and acting the Laplacian Δ on (2.12), we have

$$(trA^2 + \lambda)(3trA^2 + 2\alpha) = -\frac{3}{2}\langle gradH, gradH\rangle. \tag{2.13}$$

(2.5),(2.6) and (2.11) imply that

$$\lambda c = \lambda c^T + \lambda\langle c, \nu\rangle\nu = gradH + (trA^2 + \lambda)\nu.$$

So we have

$$\langle gradH, gradH\rangle = \langle\lambda c, \lambda c\rangle - (trA^2 + \lambda)^2.$$

Replacing this in (2.13) we have

$$(trA^2 + \lambda)(3trA^2 + 2\alpha) = \frac{3}{2}(trA^2 + \lambda)^2 - \frac{3}{2}\langle\lambda c, \lambda c\rangle.$$

This and (2.12) imply that H and trA^2 are constants. So we proved that if M is of null 2-type, then M has constant mean curvature and constant scalar curvature. We will proceed under the assumption that M is not of null 2-type and will get a contradiction. From (2.8) and (2.9) we have

$$x = \beta H\nu + x^T, \tag{2.14}$$

$$a\langle x, c^T\rangle = -\beta\Delta H - \beta H trA^2 - H, \tag{2.15}$$

where $\beta = \frac{1}{\lambda_1\lambda_2}(\lambda_1 + \lambda_2 - a)$. From (2.14) the following holds

$$a\langle x, c\rangle = \beta\Delta H + a\langle x^T, c\rangle.$$

Since $H = a\langle x, c\rangle + b$, from this we have

$$a\langle x^T, c\rangle = H - b - \beta\Delta H.$$

From this and (2.15) we have

$$trA^2 = \frac{1}{\beta}(\frac{b}{H} - 2).$$

This implies that

$$grad(trA^2) = \frac{b}{H^2\beta} gradH. \tag{2.16}$$

Since we have already seen that $grad(trA^2) = -A(gradH)$, (2.16) implies that

$$A(gradH) = -\frac{b}{H^2\beta} gradH. \tag{2.17}$$

On the other hand, the equation $\langle x, \nu \rangle = \beta H$ implies that $\beta gradH = -Ax^T$. (2.4),(2.17) and this imply that

$$2(\frac{b}{H^2\beta})^2 gradH + \frac{b}{H\beta} gradH = -\lambda_1\lambda_2\beta gradH.$$

This equation implies that H is constant. The constancy of H and (2.1) imply that M is of null 2-type, which is a contradiction to our assumption. The converse is an easy computation.

References

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