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ON A GENERALIZED ALMOST KAEHLERIAN FINSLER MANIFOLD

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1. Introduction

Let M be a $2n$-dimensional differential manifold admitting an almost complex structure $f^i{}_j(x)$ and a Finsler metric $g_{ij}(x,y)$ given by

$$g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2(x,y), \tag{1.1}$$

where $\dot{\partial}_i = \partial/\partial y^i$.

If the fundamental function $L(x,y)$ satisfies the so called Rizza condition, that is,

$$L(x, \phi_\theta y) = L(x,y) \tag{1.2}$$

for any $\theta \in R$, where

$$\phi_\theta{}^i{}_j = cos\theta \cdot \delta^i_j + sin\theta \cdot f^i{}_j,$$

then M is called an almost Hermitian Finsler manifold or simply a Rizza manifold. The almost Hermitian Finsler structure $(f^i{}_j(x), g_{ij}(x,y))$ was introduced by G. B. Rizza [5]. Afterword, it was studied by some authors. In [1] M. Fukui has proved that if $g_{ij}(x,y)$ and $f^i{}_j(x)$ satisfies the condition

$$g_{ij}(x,y) - g_{pq}(x,y)f^p{}_i(x)f^q{}_j(x) = 0,$$

then g_{ij} is a Riemannian metric, that is, $(f^i{}_j, g_{ij})$ is an almost Hermitian structure. In [2] it is known that the Rizza condition (1.2) is equivalent to any one of the following

(1) $g_{pq}(x,\phi_\theta y)\phi_\theta{}^p{}_i\phi_\theta{}^q{}_j = g_{ij}(x,y)$,
(2) $g_{ij}(x,y)f^i{}_k(x)y^k y^j = 0$,
(3) $(g_{im}(x,y) - g_{pq}f^p{}_i(x)f^q{}_m(x))y^m = 0$,
(4) $g_{im}(x,y)f^m{}_j(x) + g_{jm}(x,y)f^m{}_i(x) + 2C_{ijm}(x,y)f^m{}_r(x)y^r = 0$.

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We define in the present paper a generalized Finsler connection $\tilde{\Gamma}$ and a generalized almost Kaehlerian Finsler manifold with respect to $\tilde{\Gamma}$.

The purpose of the preset paper is to study the generalized almost Kaehlerian Finsler manifold.

Throughout the present paper we shall use the terminology and notations in Matsumoto's monograph [3].

2. A Generalized Finsler connection

Let M be an almost Hermitian Finsler manifold with the Rizza structure $(f^i{}_j(x), g_{ij}(x,y))$. If we put

$$(2.1) \qquad \tilde{g}_{ij} = \frac{1}{2}(g_{ij}(x,y) + g_{pq}(x,y)f^p{}_i(x)f^q{}_j(x)),$$

then $\tilde{g}_{ij}$ is a homogeneous symmetric Finsler metric, which is called a generalized Finsler metric induced from the Finsler metric g_{ij}. It should easily verified that

$$\tilde{g}_{ij}(x,y) = \tilde{g}_{pq}(x,y)f^p{}_i(x)f^q{}_j(x),$$

from which

$$(2.2) \qquad \tilde{g}_{im}(x,y)f^m{}_j(x) = -\tilde{g}_{jm}(x,y)f^m{}_i(x).$$

Concerning the reciprocal tensor $\tilde{g}^{ij}(x,y)$ of $\tilde{g}_{ij}(x,y)$, we could prove

$$(2.3) \qquad \tilde{g}^{ij}(x,y) = \tilde{g}^{kl}(x,y)f^i{}_k(x)f^j{}_l(x)$$

from which

$$\tilde{g}^{ik}(x,y)f^j{}_k(x) = -\tilde{g}^{jk}(x,y)f^i{}_k(x).$$

Now we put

$$(2.4) \qquad f_{ij}(x,y) = g_{im}(x,y)f^m{}_j,$$

$$(2.5) \qquad \tilde{f}_{ij}(x,y) = \tilde{g}_{im}(x,y)f^m{}_j.$$

By virtue of (2.2) we have

$$\tilde{f}_{ij}(x,y) = -\tilde{f}_{ji}(x,y), \qquad \tilde{f}_{im}(x,y)f^m{}_j(x) = -\tilde{g}_{ij}(x,y), \tag{2.6}$$
$$\tilde{f}_{ij}(x,y) = \frac{1}{2}(f_{ij}(x,y) - f_{ji}(x,y)).$$

Let us consider such connection that

$$\tilde{\Gamma}_j{}^i{}_k = \frac{1}{2}\tilde{g}^{im}(X_k\tilde{g}_{jm} + X_j\tilde{g}_{mk} - X_m\tilde{g}_{kj}), \tag{2.7}$$

where $X_k = \partial_k - N^l{}_k(x,y)\dot{\partial}_l$, $N^l{}_k$ is a non linear connection and $\partial_k = \partial/\partial x$. Then $\tilde{\Gamma}_j{}^i{}_k$ is symmetric and satisfies the transformation rule of a linear connection. So, we represent by $\tilde{\nabla}_k$ the h-covariant derivative with respect to $(\tilde{\Gamma}_j{}^i{}_k, N^i{}_j)$.

For any Finsler tensor $T^i{}_j(x,y)$ of (1,1)-type, the h-covariant derivative with respect to $(\tilde{\Gamma}_j{}^i{}_k, N^i{}_j)$ are expressed as follows:

$$\tilde{\nabla}_k T^i{}_j = \partial_k T^i{}_j - N^l{}_k\dot{\partial}_l T^i{}_j + \tilde{\Gamma}_r{}^i{}_k T^r{}_j - T^i{}_r\tilde{\Gamma}_j{}^r{}_k.$$

Therefore, for the almost complex structure tensor $f^i{}_j(x)$

$$\tilde{\nabla}_k f^i{}_j(x) = \partial_k f^i{}_j(x) + \tilde{\Gamma}_r{}^i{}_k f^r{}_j(x) - f^i{}_r(x)\tilde{\Gamma}_j{}^r{}_k. \tag{2.9}$$

Using (2.7) we obtain

$$\tilde{\nabla}_k\tilde{g}_{ij} = \partial_k\tilde{g}_{ij} - N^l{}_k\dot{\partial}_l\tilde{g}_{ij} - X_k\tilde{g}_{ij} = 0. \tag{2.10}$$

Thus we have

THEOREM 2.1. *A Finsler space with a generalized Hermitian structure* $(\tilde{\Gamma}_j{}^i{}_k, N^i{}_j)$ *is h-metrical*

3. A generalized almost Kaehlerian Finsler manifold

A generalized Hermitian Finsler manifold M with a $(f^i{}_j(x), \tilde{g}_{ij}(x,y), N)$-struc- ture satisfying $\tilde{\nabla} f^i{}_j = 0$ is called a generalized Kaehlerian

Finsler manifold [3], and M satisfying $\tilde{\nabla}_k f^i{}_j + \tilde{\nabla}_j f^i{}_k = 0$ is said a generalized nearly Kaehlerian Finsler manifold.

Now, in a generalized Hermitian Finsler manifold M with $(f^i{}_j(x), \tilde{g}_{ij}(x,y), N)$-structure, we put

$$\tilde{F}_{ijk} + X_i \tilde{f}_{jk} + X_j \tilde{f}_{ki} + X_k \tilde{f}_{ij},$$

then from (2.6) we have

$$\tilde{F}_{ijk} = \tilde{\nabla}_i \tilde{f}_{jk} + \tilde{\nabla}_j \tilde{f}_{ki} + \tilde{\nabla}_k \tilde{f}_{ij}. \tag{3.1}$$

A generalized Hermitian Finsler manifold M with a $(f^i{}_j(x), \tilde{g}_{ij}(x,y), N)$-structure satisfying $\tilde{F}_{ijk} = 0$ is called a generalized almost Kaehlerian Finsler manifold, which following the example of complex Riemannian geometry.

On the other hand the Nijenhuis tensor $N^i{}_{jk}$ of almost complex structure $f^i{}_j(x)$ is defined as follos [7]:

$$N^i{}_{jk} = (\partial_r f^i{}_j) f^r{}_k - (\partial_r f^i{}_k) f^r{}_j + f^i{}_r \partial_j f^r{}_k - f^i{}_r \partial_k f^r{}_j.$$

Substituting (2.9) in the above equation we have

$$\begin{aligned} N^i{}_{jk} =& (\tilde{\nabla}_r f^i{}_j - \tilde{\Gamma}_m{}^i{}_r f^m{}_j + f^i{}_m \tilde{\Gamma}_j{}^m{}_r) f^r{}_k \\ & - (\tilde{\nabla}_r f^i{}_k - \tilde{\Gamma}_m{}^i{}_r f^m{}_k + f^i{}_m \tilde{\Gamma}_k{}^m{}_r) f^r{}_j \\ & + f^i{}_r (\tilde{\nabla}_j f^r{}_k - \tilde{\Gamma}_m{}^r{}_j f^m{}_k - f^r{}_m \tilde{\Gamma}_k{}^m{}_j) \\ & - f^i{}_r (\tilde{\nabla}_k f^r{}_j - \tilde{\Gamma}_m{}^r{}_k f^m{}_j - f^r{}_m \tilde{\Gamma}_j{}^m{}_k) \\ =& (\tilde{\nabla}_r f^i{}_j) f^r{}_k - (\tilde{\nabla}_r f^i{}_k) f^r{}_j + f^i{}_r \tilde{\nabla}_j f^r{}_k - f^i{}_r \tilde{\nabla}_k f^r{}_j. \end{aligned} \tag{3.2}$$

Moreover let us put $\tilde{N}_{hij} = \tilde{g}_{hm} N^m{}_{ij}$. Then we have

$$\tilde{N}_{hij} = (\tilde{\nabla}_r \tilde{f}_{hi}) f^r{}_j - (\tilde{\nabla}_r \tilde{f}_{hj}) f^r{}_i + \tilde{f}_{hr} \tilde{\nabla}_i f^r{}_j - \tilde{f}_{hr} \tilde{\nabla}_j f^r{}_i \tag{3.3}$$

by virtue of (2.5) and (2.10).

From (2.6) ,(3.1) and $(\tilde{\nabla}_i \tilde{f}_{hr}) f^r{}_j = \tilde{f}_{hr} \tilde{\nabla}_i f^r{}_j$, (3.3) is reduced to

$$\tilde{N}_{hij} = f^r{}_j \tilde{F}_{rhi} - f^r{}_i \tilde{F}_{rhj} - 2\tilde{f}_{jr} \tilde{\nabla}_h f^r{}_i. \tag{3.4}$$

Since $\tilde{F}_{ijk} = 0$ in a generalized almost Kaehlerian Finsler manifold, we have

$$\tilde{N}_{hij} = -2\tilde{g}_{jm} f^m{}_r \tilde{\nabla}_h f^r{}_i.$$

Thus we have

THEOREM 3.1. *A generalized almost Kaehlerian Finsler manifold is a generalized Kaehlerian Finsler manifold if and only if* $\tilde{N}_{hji} = 0$

From (3.4) we have

$$\tilde{N}_{hij} + \tilde{N}_{ihj} = -f^r{}_i\tilde{F}_{rhj} - f^r{}_h\tilde{F}_{rij} - 2\tilde{f}_{jr}(\tilde{\nabla}_h f^r{}_i + \tilde{\nabla}_i f^r{}_h).$$

In a generalized almost Kaehlerian Finsler manifold we get

$$\tilde{N}_{hij} + \tilde{N}_{ihj} = -\tilde{g}_{jm}f^m{}_r(\tilde{\nabla}_h f^r{}_i + \tilde{\nabla}_i f^r{}_h).$$

Thus we have

THEOREM 3.2. *A generalized almost Kaehlerian Finsler manifold is a generalized nearly Kaehlerian Finsler manifold if and only if* $\tilde{N}_{hij} + \tilde{N}_{ihj} = 0$.

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