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ON THE EXTREMAL PROBLEMS IN THE PLANE

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1. Introduction.

In general, extremal problems are problems in which it is required to find the precise upper or lower limit of a functional defined with respect to a given class of functions

The Riemann Mapping Theorem [1,4,5], in its most common form, asserts that any two simply connected proper subdomains of the plane are conformally equivalent. More precisely, the Riemann Mapping Theorem says that for any simply connected domain Ω and $a \in \Omega$, there exists a conformal mapping ϕ of Ω onto the unit disc Δ such that $\phi(a) = 0$, $\phi'(a) > 0$. This Riemann Mapping Function is the unique solution to an extremal problem with respect to the family of holomorphic mappings on Ω.

In the present work, we study some extremal problems in the plane. In section 2, we will introduce some necessary ingredients to prove the extremal problems in section 3 and 4. Section 3 is devoted to prove a higher order extremal problem on simply connected domains in the plane. In section 4, we investigate an extremal problem in multiply connected domains.

2. Preliminaries.

In this section, we introduce some theorems needed in section 3 and 4. The following lemma can be proved using the Maximum modulus theorem.

Lemma 2.1. *Let $|\alpha| < 1$. The map ϕ_α defined by $\phi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha}z)$ is a holomorphic automorphism of unit disc Δ and $\phi'_\alpha(z) = (1 - |\alpha|^2)/(1 - \bar{\alpha}z)^2$.*

This map ϕ_α can be powerfully used, in variations of Schwarz lemma, to solve various extremal problems for holomorphic mappings.

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Let $H(\Omega)$ be the class of all holomorphic mappings in Ω. Suppose $\mathcal{N} \subset H(\Omega)$. We call $\mathcal{N}$ a normal family if every sequence of members of $\mathcal{N}$ contains a subsequence which converges uniformly on compact subsets of Ω.

THEOREM 2.2. *Suppose $\mathcal{N} \subset H(\Omega)$ and $\mathcal{N}$ is uniformly bounded on each compact subset of the region Ω. Then $\mathcal{N}$ is normal family.*

Proof. . See [1, pp. 282].

THEOREM 2.3. *Suppose $f_n \in H(\Omega)$, for $n = 1, 2, 3 \cdots$, and $f_n \longrightarrow f$ uniformly on compact subsets of Ω. Then $f \in H(\Omega)$, and $f_n' \longrightarrow f'$ uniformly on compact subsets of Ω.*

Proof. See [2, pp. 214].

If we apply Theorem 2.3 sucessively, we get the following corollary.

COROLLARY 2.4. *Under the same hypothesis, $f_n^{(k)} \longrightarrow f^{(k)}$ uniformly, as $n \longrightarrow \infty$, on every compact sets of Ω, and for every positive integer k.*

HURWITZ'S THEOREM. *If the mappings $f_n(z)$ are holomorphic and never zero in a region Ω, and if $f_n(z)$ converges to $f(z)$, unifomly on every compact subset of Ω, then $f(z)$ is either identically zero or never equal to zero in Ω.*

Note that $dz \wedge d\bar{z} = -2i\ dx \wedge dy$. So we have the following complex version of Green's formula.

GREEN'S FORMULA. *Suppose that $f(z)$and $g(z)$ are holomorphic function in a domain Ω and on its piecewise smooth boundary C. Then*

$$\iint_{\Omega} f(z)\overline{g'(z)}\, dxdy = \frac{1}{2i} \int_C f(z)\overline{g(z)}dz.$$

3. Extremal problems on simply connected domains. .

In this section, we will study the extremal problem of higher order on simply connected domains. In [2,3], Chung considered 2nd order version of extremal problems on bounded n-connected domains in the plane. He got some relations between the Ahlfors mappings and all holomorphic mappings on the multiply connected domains in the plane. First, we show that the Riemann Mapping Function is the unique solution to some of the extremal problem.

THEOREM 3.1. *Suppose Ω is a simply connected domain and $a \in \Omega$ is given. Let $\mathcal{F}$ be the class of all holomorphic mappings h of Ω ($\neq \mathbb{C}^1$) into the unit disc Δ. Then the Riemann Mapping Function associated to a point $a \in \Omega$ is the unique solution to the extremal problem :*

$$f \in \mathcal{F}; \quad f'(a) = \sup\{|h'(a)| : \quad h \in \mathcal{F}\}.$$

Proof . Let $M = \sup\{|h'(a)| : h \in \mathcal{F}\}$. Then there exists a sequence $\{h_n\}$ in $\mathcal{F}$ such that $|h_n'(a)| \longrightarrow M$. Since $|h(z)| < 1$ for all $h \in \mathcal{F}$ and $z \in \Omega$, Theorem 2.2 shows that $\mathcal{F}$ is a normal family. Thus we can extract a subsequence (again denoted by $\{h_n\}$ for simplicity) which converges, uniformly on compact subsets of Ω, to a limit $g \in H(\Omega)$. By Theorem 2.3, h_n' converges to g', uniformly on compact subsets of Ω, and hence $|g'(a)| = M$. We may assume that $g'(a) = M$ because we can multiply $e^{i\theta}$ if it is necessary. Let us now show that $g \in \mathcal{F}$. Since $\mathcal{F}$ contains the Riemann mapping Function f, $g'(a) \geq f'(a) > 0$ and hence g is not a constant function. Since $h_n(\Omega) \subset \Delta$, for $i = 1.2.3. \cdots$, we have $g(\Omega) \subset \bar{\Delta}$. But by Open mapping theorem, $g(\Omega)$ is an open set and hence $g(\Omega) \subset \Delta$. Next let us show that $g(a) = 0$. Suppose, on the contrary, that $g(a) = c$ ($\neq 0$). Set $\phi_c = (z-c)/(1-\bar{c}z)$ and consider the sequence of mappings $\{\phi_c \circ h_n\}$ on Ω. Then

$$\begin{aligned} \lim_{n\infty}(\phi_c \circ h_n)'(a) &= \lim_{n\infty} \phi_c'(h_n(a))h_n'(a) = \phi_c'(g(a))g'(a) \\ &= \frac{1-|c|^2}{(1-\bar{c}g(a))^2}g'(a) = \frac{M}{1-|c|^2} > M. \end{aligned} \tag{3.1}$$

Since $(\phi_c \circ h_n) \in \mathcal{F}$ for all n, (3.1) shows that $\sup\{|h'(a)| : h \in \mathcal{F}\} > M$, which is a contradiction. Consequently $g \in \mathcal{F}$, $g'(a) = \sup\{|h'(a)| : h \in \mathcal{F}\}$ and $g(a) = 0$. Now let f be the Riemann Mapping Function corresponding a point $a \in \Omega$. To show $f = g$, we claim that f is a solution of the extremal problem. Let $h \in \mathcal{F}$. If $h(a) \neq 0$, we consider $\psi = \phi_{h(a)} \circ h \circ f^{-1} : \Delta \longrightarrow \Delta$, where $\phi_{h(a)}(z) = (z - h(a))/(1 - \overline{h(a)}z)$.

Then ψ is holomorphic in Δ and $\psi(0) = 0$. By Schwarz lemma, we have $|\psi'(0)| \leq 1$. But

$$\begin{aligned} |\psi'(0)| &= \left| \phi'_{h(a)}(h(a))h'(a)\frac{1}{f'(f^{-1}(0))} \right| \\ &= \left| \frac{1}{1-|h(a)|^2} h'(a) \frac{1}{f'(a)} \right| \leq 1 \quad , \end{aligned} \tag{3.2}$$

and hence $|h'(a)| < |f'(a)|$. This shows that $f'(a) = \sup\{|h'(a)| : h \in \mathcal{F}\}$. Since g is the solution to the extremal problem, $f'(a) = g'(a)$. It remains to show that g is identically equal to f in Ω. Consider the composite mapping $\varphi = g \circ f^{-1} : \Delta \longrightarrow \Delta$. Then $\varphi(0) = 0$ and $\varphi'(0) = (g \circ f^{-1})'(0) = g'(a)/f'(a) = 1$. By Schwarz lemma, we have $\varphi(z) = e^{i\theta} z$. If $z = f(w)$, then $g(w) = g(f^{-1}(z)) = \varphi(z) = e^{i\theta} f(w)$ and hence $g'(a) = e^{i\theta} f'(a) > 0$, so that $\theta = 0$. Therefore $g \equiv f$ in Ω.

Now we consider the higher order version of the extremal problems. First, we prove the following lemma.

LEMMA 3.2. *Let $k \geq 1$ be an integer and let H_k be the class of holomorphic mappings h of the unit disk Δ into itself with $h(0) = h'(0) = \cdots = h^{(k-1)}(0) = 0$. Then the mapping $g(z) = z^k$ is the unique solution of the extremal problem :*

$$g \in H_k; \quad g^{(k)}(0) = \sup\{|h^{(k)}(0)| : \quad h \in H_k\}.$$

Proof. Let $M = \sup\{|h^{(k)}(0)| : h \in H_k\}$. Then there exist a sequence $\{h_n\}$ in H_k such that $|h_n^{(k)}(0)| \longrightarrow M$. By the normal property of H_k (Theorem 2.2), we obtain a subsequence $\{h_n\}$ in H_k which converges, uniformly on compact subsets of Δ, to a holomorphic mapping g, and hence $h_n^{(l)} \longrightarrow g^{(l)}$ $l = 1, 2, \cdots, k$, uniformly on compact subsets of Δ by Corollary 2.4. Hence $g^{(k)}(0) = M$. Since $h_n^{(l)}(0) = 0$ for $l = 0, 1, 2, \cdots k-1$, we have $g(0) = g'(0) = \cdots = g^{(k-1)}(0) = 0$. Consequently $g \in H_k$ is a solution to the extremal problem. We claim that $g(z) = z^k$. Let us prove this lemma by induction on k. If $k = 1$, then $g'(0) = 1 = \sup\{|h'(0)| : h \in H_1\}$ by Schwarz lemma. Suppose the lemma has been established for $k-1$ $(k > 1)$, and suppose that $g \in H_k$ is an extremal mapping. Consider the mapping

$\phi(z) = g(z)/z$ on Δ. Then $\phi(z)$ is of the form $z^{k-1}p(z)$, $p \in H(\Delta)$ and so $\phi(0) = \phi'(0) = \cdots = \phi^{(k-2)}(0) = 0$. Since $g(0) = 0$ and $|g(z)| \leq 1$, we have $|\phi(z)| \leq 1$ by Schwarz lemma. Hence it follows that $\phi \in H_{k-1}$. To show that $\phi^{(k-1)}(0) = \sup\{|h^{(k-1)}(0)| : h \in H_{k-1}\}$, we let $h \in H_{k-1}$ and consider a mapping $\psi(z) = zh(z)$. Then $\psi(\Delta) \subset \Delta$, and $\psi(0) = \psi'(0) = \cdots = \psi^{(k-1)}(0) = 0$, so that $\psi(z)$ is in H_k. By the extremal property of g, $|\psi^{(k)}(0)| \leq |g^{(k)}(0)|$. Note that $g'(z) = k\phi(z) + z^k p'(z)$. Thus $g^k(z) = k\phi^{k-1}(z) + zq(z)$ for some $q \in H(\Delta)$, and so $\phi^{(k-1)}(0) = g^{(k)}(0)/k$. Similarly, we have $\psi^{(k)}(z) = kh^{(k-1)}(z) + zh^k(z)$. Hence it follows that

$$|h^{(k-1)}(0)| = \left|\frac{\psi^{(k)}(0)}{k}\right| \leq \left|\frac{g^{(k)}(0)}{k}\right| = \left|\phi^{(k-1)}(0)\right|.$$

By our induction hypothesis, we obtain

$$\phi(z) = g(z)/z = z^{k-1}, \qquad or \qquad g(z) = z^k,$$

which complete the proof of the lemma.

Using Lemma 3.2 we can prove the following generalized version of higher order extremal problems.

THEOREM 3.3. *Suppose that Ω is a simply connected domain and $a \in \Omega$ is given. Let $k \geq 1$ be an integer and let $\mathcal{F}$ be the class of holomorphic mappings h of Ω into the unit disc Δ with $h(a) = h'(a) = \cdots = h^{(k-1)}(a) = 0$. Then the power $g = f^k$ of the Riemann Mapping Function f associated to a point $a \in \Omega$ is the unique solution of the extremal problem:*

$$g \in \mathcal{F}; \quad g^{(k)}(a) = \sup\{|h^{(k)}(a)| : \quad h \in \mathcal{F}\}.$$

Proof. . Using the fact that $\mathcal{F}$ is normal, we can show that there exists an extremal mapping for the family $\mathcal{F}$, as we did in the beginning of the proof of Theorem 3.1. Suppose that $g \in \mathcal{F}$ is a solution to the extremal problem. Let ϕ be a holomorphic mapping of unit disc Δ into itself with $\phi(0) = \phi'(0) = \cdots = \phi^{(k-1)}(0) = 0$. Consider the composite mapping $h = \phi \circ f$ of Ω into Δ. Then $h \in \mathcal{F}$, since

$h(a) = h'(a) = \cdots = h^{(k-1)}(a) = 0$. By the extremal property of g, $|h^{(k)}(a)| \le |g^{(k)}(a)|$. Consider the composite mapping $\psi = g \circ f^{-1}$ which is a holomorphic mapping of the unit disc Δ into itself. Then, since $\phi = h \circ f^{-1}$ and $\phi^{(l)}(0) = 0$ for $l = 1, 2, \cdots, k-1$, we have

$$|\phi^{(k)}(0)| = \left| \frac{h^{(k)}(a)}{(f'(a))^k} \right| \le \left| \frac{g^{(k)}(a)}{(f'(a))^k} \right| = |\psi^{(k)}(0)|.$$

Hence ψ is the unique solution to the extremal problem with respect to the the family H_k. By Lemma 3.2, we obtain

$$\psi(z) = z^k$$

Therefore $g(z) = (f(z))^k$. The proof of theorem is completed.

4. An extremal problem on multiply connected domains.

In this section, we consider an extremal problem for multiply connected domains in the plane.

DEFINITION 4.1. *A domain Ω is said to have the finite connectivity n if the complement of Ω has exactly n components and infinite connectivity if the complement has infinitely many components*

DEFINITION 4.2. *A real or complex function $\varphi(t)$ of a real variable t, defined on an interval $a < t < b$, is said to be real analytic (or analytic in the real sense) if, for every t_0 in the interval, the Taylor development $\varphi(t) = \varphi(t_0) + \varphi'(t_0)(t - t_0) + \frac{1}{2}\varphi''(t_0)(t - t_0)^2 + \cdots$, converges in some interval $(t_0 - \rho, \quad t_0 + \rho)$, $\rho > 0$.*

THEOREM 4.3. *An arbitary domain Ω of connectivity n is holomorphically equivalent to a domain $\widetilde{\Omega}$ which is bounded by n analytic closed curves.*

Proof. . We assume, in order to exclude trival cases, that each of the n boundary components $C_1, C_2, \cdots, C_n$ consists of more than one boundary point. We denote by A the domain bounded by C_1 which is obtained from Ω by filling in the holes bounded by $C_2, C_3, \cdots, C_n$. (if Ω is finite and C_1 is not outer boundary of Ω, one of these holes will contain the point at infinity). By Riemann Mapping Theorem, A may be mapped conformally onto the interior of the unit circle. By this mapping, Ω is transformed into a domain Ω_1 which is bounded by the unit circle and by $n-1$ continua, say ${C_2}', {C_3}', \cdots, {C_n}'$, which are the conformal images of $C_2, C_3, \cdots, C_n$. Next, denote by B the simply connected domain with respect to the extended plane, bounded by ${C_2}'$ which is obtained by filling in all the other holes of Ω_1 (including the exterior of the unit circle). This domain may again be mapped onto the interior of the unit circle, and this mapping will transform Ω_1 into a domain Ω_2 which is bounded by the unit circle and by $n-1$ other continua. One of these continua is the conformal image of the unit circle as yielded by an holomophic function which is regular at all points of the unit circle (this circle is in the interior of B). Hence, this continuum is a closed analytic curve. Two of the boundary components of Ω_2, namely, this curve and the unit circle, are therefore closed analytic curves. Continuing in this fashion, we arrive after another $n-2$ auxiliary mappings at a conformal map of Ω which is bounded by n closed analytic curves.

COROLLARY 4.4. *Let Ω be a n-connected domain in the plane. Then for each $a \in \Omega$, there is a biholomorphism $\Phi_a : \Omega \longrightarrow \widetilde{\Omega}$, such that $\Phi'(a) > 0$, where $\widetilde{\Omega}$ is bounded by n closed analytic curves.*

proof. . From Theorem 4.3, there exists a biholomorphism, $\Phi : \Omega \longrightarrow \Omega_1$, where Ω_1 is a domain having n analytic boundaries. Since $\Phi'(a) \neq 0$, there is a θ such that $e^{i\theta}\Phi'(a) > 0$. Set $\Phi_a = e^{i\theta}\Phi$. Then $\Phi_a'(a) = e^{i\theta}\Phi'(a) > 0$, and $\widetilde{\Omega} = \Phi_a(\Omega)$ has n analytic boundaries.

LEMMA 4.5. *Let $p = p(z)$ and $q = q(z)$ be two holomorphic functions which have the following properties: p and q are single-valued in a multiply connected domain Ω which is bounded by n analytic curves C; p and q are holomorphic in Ω and on its boundary C with the possible exception of a finite number of poles in Ω, while $p - q$ is holomorphic in Ω and on C. Then*

$$\iint_\Omega |p' - q'|^2 dxdy = Re\left[\frac{1}{i}\int_C (\bar{p} - \bar{q})p' dz\right] + \frac{1}{2i}\int_C \bar{q}q'\, dz - \overline{\frac{1}{2i}\int_C \bar{p}p' dz}$$

Proof. Since $p-q$ is holomorphic and single-valued in Ω, it follows from Green's formula that

$$\begin{aligned}\iint_\Omega |p'-q'|^2\,dxdy &= \iint_\Omega (p'-q')(\overline{p'}-\overline{q'})\,dxdy\\ &= \frac{1}{2i}\int_C (\bar p-\bar q)(p'-q')\,dz\\ &= \frac{1}{2i}\int_C (\bar p-\bar q)p'\,dz + \frac{1}{2i}\int_C \bar q q'\,dz - \frac{1}{2i}\int_C \bar p q'\,dz.\end{aligned} \tag{4.1}$$

The last line integral can be transformed by integration by parts. We have

$$\begin{aligned}-\frac{1}{2i}\int_C \bar p q'\,dz = -\frac{1}{2i}\int_C \bar p\,dq &= -\frac{1}{2i}[\bar p q]_C + \frac{1}{2i}\int_C q\,\overline{dp}\\ &= -\frac{1}{2i}[\bar p q]_C - \overline{\frac{1}{2i}\int_C \bar q p'\,dz},\end{aligned}$$

where $[\bar p q]_C$ denotes the integrated part. Notice that the integrated part vanishes. Indeed, integration is carried out, in turn, over the closed curves $C_1, C_2, C_3, \cdots, C_n$. Since p and q are single-valued in Ω, the expression $\bar p q$ returns to its initial value if z describes any of these closed curves. Hence,

$$-\frac{1}{2i}\int_C \bar p q'\,dz = -\overline{\frac{1}{2i}\int_C \bar q p'\,dz} = \overline{\frac{1}{2i}\int_C (\bar p-\bar q)p'\,dz} - \overline{\frac{1}{2i}\int_C \bar p p'\,dz}.$$

Inserting this relation in (4.1), we obtain the identity.

We now ready to solve the following extremal problem for the multiply connected domains in the plane.

THEOREM 4.6. *Let Ω denote a bounded n-connected domain in the plane and let $\mathcal{F}$ denote the class of all one-to-one holomorphic mappings from Ω into the unit disc Δ. Given a point $a \in \Omega$, suppose that f is a function in $\mathcal{F}$ such that $f(a)=0$, $f'(a)$ is real and positive, and $f(\Omega)$ is equal to the unit disc minus $n-1$ disjoint circular slits. Then the function f is the unique solution to the extremal problem:*

$$f \in \mathcal{F}; \quad f'(a) = \sup\{|h'(a)| : \quad h \in \mathcal{F}\}.$$

Proof. By the normal property of $\mathcal{F}$ and Theorem 2.3, we obtain a sequence $\{h_n\}$ in $\mathcal{F}$ which $h_n \longrightarrow g \in H(\Omega)$, and hence $h_n' \longrightarrow g'$, uniformly on compact subsets of Ω and $g'(a) = \sup\{|h'(a)|\ h \in \mathcal{F}\}$. Since $\mathcal{F}$ contains f, $g'(a) \geq f'(a) > 0$. Let us now show that $g \in \mathcal{F}$. Fix $z_1 \in \Omega$. Let $p_n(z) = h_n(z) - h_n(z_1)$ and $p = g(z) - g(z_1)$. Then $p_n(z)$ is never zero on $\Omega - \{z_1\}$ and p_n converges to p, uniformly on compact subsets of Ω. Since g is nonconstant, p is not identically to zero. By Hurwitz's theorem, p is never zero on $\Omega - \{z_1\}$. That is, if $z_2 \in \Omega - \{z_1\}$, then $g(z_1) \neq g(z_2)$. Hence g is one to one. Moreover we have $g(a) = 0$ by using the exactly same method as the proof of Theorem 3.1. Before taking concrete shape to our proof, we intend to replace a given domain Ω by a domain $\widetilde{\Omega}$ which is bounded by n closed analytic curves. Let us show that it is possible. First, we assume that the following statement $(*)$ is true.

$(*)$ *Let $\widetilde{\Omega}$ be a bounded n-connected domain in plane with analytic closed boundaries and let $\widetilde{\mathcal{F}}$ be the class of all one-to-one holomorphic mappings from $\widetilde{\Omega}$ into the unit discΔ. Given a point $\widetilde{a} \in \widetilde{\Omega}$, suppose that $\widetilde{f}$ is a function in $\widetilde{\mathcal{F}}$ such that $\widetilde{f}(\widetilde{a}) = 0$, $\widetilde{f}'(\widetilde{a})$ is real and positive, and $\widetilde{f}(\widetilde{\Omega})$ is equal to the unit disc minus $n-1$ disjoint circular slits. Then $\widetilde{f}'(\widetilde{a}) = \sup\{|\widetilde{h}'(\widetilde{a})| : \widetilde{h} \in \widetilde{\mathcal{F}}\}$.*

Let $\Phi : \Omega \longrightarrow \widetilde{\Omega}$ be a biholomorphism such that $\Phi(a) = \widetilde{a}$ and $\Phi'(a) > 0$, as in corollary 4.4, where $\widetilde{\Omega}$ has analytic boundaries. Also set $\widetilde{\mathcal{F}} = \{h \circ \phi^{-1} : h \in \mathcal{F}\}$. Then it can be easily shown that $\widetilde{\mathcal{F}}$, $\widetilde{f} = f \circ \phi^{-1}$, and $\widetilde{\Omega}$ satisfy all the conditions in the statement $(*)$. Hence it follows that

$$\begin{aligned} f'(a) &= \widetilde{f}'(\Phi(a))\Phi'(a) = \Phi'(a)\sup\{|\widetilde{h}'(a)| : \widetilde{h} \in \mathcal{F}\} \\ &= \Phi'(a)\sup\{|h'(a)(\Phi'(a))^{-1}| : h \in \mathcal{F}\} = \sup\{|h'(a)| : h \in \mathcal{F}\}. \end{aligned}$$

Thus we may replace the given Ω by the domain with analytic boundaries. We claim that $g'(a) \leq f'(a)$. First, we consider the function f which yields the circular slit mapping, and the extremal mapping g which we have already obtained in the opening part of the proof. The function $\log f$ and $\log g$ are not holomorphic in Ω, since they have logarithmic poles at $z = a$. Although not single-valued in Ω, these functions have no periods about the boundary components C_k of the domain Ω, that is, $\int_{C_k} d[\log f] = \int_{C_k} d[\log g] = 0$. Indeed,

$$\int_{C_k} d[\log f] = \int_{C_k} d[\log |f|] + i \int_{C_k} d[\arg f].$$

Since $\log |f|$ is single-valued in Ω, the first integral on the right-hand side is zero. the second integral vanishes since the conformal image of C_k dose not surround the orign, and therefore $\arg f$ returns to its initial value. The result for $\log g$ follows in the same way. The function $\log\left(\frac{f}{g}\right)$ is holomorphic and single-valued in Ω. Indeed, the singularities of $\log f$ and $\log g$ cancel each other, and $\log(\frac{f}{g})$ is free of period about the boundary components, since the same is true of $\log f$, $\log g$. By Lemma 4.5 we obtain

$$\iint_{\Omega} \left|\frac{f'}{f} - \frac{g'}{g}\right|^2 dxdy = Re\left[\frac{1}{i}\int_c \log\left(\frac{f}{g}\right)\frac{f'}{f}\,dz\right] \\ + \overline{\frac{1}{2i}\int_C \overline{\log g}\frac{g'}{g}\,dz - \frac{1}{2i}\int_C \overline{\log f}\frac{f'}{f}\,dz}. \tag{4.2}$$

Since f maps the boundary components C_k onto circular slits, we have $\log |f| = Re[\log f] = \text{constant}$ if z is on C_k. Hence $d[\log f]$ is pure imaginary, that is,

$$\frac{1}{i}\frac{f'}{f}\,dz = real, \quad z \in C \quad or \quad \frac{1}{i}\frac{f'}{f}\,dz = \overline{\frac{1}{i}\frac{f'}{f}dz}. \tag{4.3}$$

In view of (4.3), the first and third line integrals on the right-hand side of (4.2) take the forms

$$Re\left[\frac{1}{i}\int_C \log\left(\frac{f}{g}\right)\frac{f'}{f}\,dz\right], \tag{4.4}$$

and

$$-\frac{1}{2i}\int_C \log f\frac{f'}{f}\,dz = -\frac{1}{4i}\int_C d[(\log f)^2], \tag{4.5}$$

respectively. The integral (4.5) vanishes since $\log f$ has no periods about the C_k. The value of (4.4) is, by the residue theorem and by the L'Hospital's theorem,

$$2\pi Re\left[\log \frac{f'(a)}{g'(a)}\right] = 2\pi \log \frac{f'(a)}{g'(a)}.$$

The integral

$$\frac{1}{2i}\int_C \overline{\log g}\frac{g'}{g}\,dz \tag{4.6}$$

can be expressed as a negative area. Indeed, since the mapping $\log g$ transforms the C_k into n simple closed curves $C_k{}'$, the integral (4.6) is equal to the negative value of the combined area A enclosed by the $C_k{}'$. Collecting our results, we thus find that

$$A + \iint_\Omega \left|\frac{f'}{f} - \frac{g'}{g}\right|^2 dxdy = 2\pi \log \frac{f'(a)}{g'(a)}.$$

The left-hand side is obviously positive unless g is identically with f. It hence follows that $g'(a) \leq f'(a)$. But clearly, $f'(a) \leq g'(a)$. Therefore $f'(a) = g'(a) = \sup\{|h'(a)| : h \in \mathcal{F}\}$. The proof of Theorem 4.3 is completed.

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