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FIBER BUNDLES THAT INDUCE APPROXIMATE FIBRATIONS

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1. Introduction.

Homotopy lifting property is a very important property for a map to have. Using the more general property of homotopy lifting property, D.S. Coram and P.F. Duvall[1] introduced the concept of an approximate fibration and showed that it has nice analogous properties to Hurewicz fibration and applying to a larger class of maps. If $p : M \to B$ is an approximate fibration to a path connected space B, then point inverses are pairwise homotopy equivalent absolute neighborhood retracts and there exists a homotopy exact sequence between M, B and fibers of p. Thus approximate fibrations are very useful concept as well as Hurewicz fibrations in the study of maps and locally prevalent.

Our principal concern is to solve the problem that under what condition a proper map is an approximate fibration. In order for a map $p : M \to B$ to be an approximate fibration, its point inverses must be pairwise homotopy equivalent. By the way, for particular integer k, any proper map defined on an $(n+k)$-manifold is always an approximate fibration owing to the fact that each point preimage is homotopy equivalent to some closed manifold N. R.J. Daverman called such manifolds N codmension k-fibrators. Unexpectedly, many collection of manifolds N has this desirable property. Note that a codimension k fibrator is a codimension $(k-1)$ fibrator, but its converse is not true, for example, the $(k-1)$-sphere $S^{(k-1)}$ is a representative example ([6],[13]).

We are not sure that the class of codimension k fibrators is closed under finite product. For a codimension 2 case, Y.H. Im ([10]) showed that the class of closed, orientable surfaces with non-zero Euler characteristic has this property. Throughout [11] and [12], Y.H. Im, M.K.

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Kang and K.M. Woo obtained the result that a product of a simply connected closed manifold and a closed aspherical manifold with hyperhopfian fundamental group is a codimension 2 fibrator. Before then, R.J. Daverman showed that each of a simply connected space ([3]) and a closed aspherical manifold with hyperhopfian fundamental group ([4]) is a codimension 2 fibrator. Our aim in this paper is to investigate codimension 2 fibrators which is closed under bundle structure.

A proper map $p : M \to B$ between locally compact ANR's is called an *approximate fibration* if it has the following approximate homotopy lifting property: given an open cover ε of B, an arbitrary space X, and two maps $g : X \to M$ and $F : X \times I \to B$ such that $p \circ g = F_0$, there exists a map $G : X \times I \to M$ such that $G_0 = g$ and $p \circ G$ is ε-close to F.

We assume all spaces are locally compact, metrizable ANR's, and all manifolds are finite dimensional, orientable, connected and boundaryless. A manifold M is said to be *closed* if M is compact, connected and boundaryless. A closed n-manifold N is called a *codimension k fibrator* if, whenever $p : M \to B$ is a proper map from an arbitrary $(n + k)$-manifold M to a finite dimensional space B such that each point preimage $p^{-1}(b)$ is homotopic equivalent to N, $p : M \to B$ is an approximate fibration.

A closed manifold N is called *Hopfian* if every degree one map $N \to N$ which induces a π_1-automorphism is a homotopy equivalence. A group H is *Hopfian* if every epimorphism $\psi : H \to H$ is necessarily an isomorphism.

A group H is said to be *residually finite* if for each $e_H \neq h \in H$, there exists a finite group A and a homomorphism $\phi : H \to A$ such that $\phi(h) \neq e_A$.

2. Semidirect product and residually finite property.

First of all, let us consider a semidirect product of groups. Recall that a group G is a semidirect product of H by K, denoted by $G = H \rtimes K$, if G contains subgroups H and K such that

(1) H is a normal subgroup;
(2) $HK = G$;
(3) $H \cap K = e_G$.

In contrast to a direct product, a semidirect product of H by K is not determined up to isomorphism by the two groups but depend on how H is normal in G. Our object is to recapture a semidirect product G from any groups H and K.

DEFINITION 2.1. *Let H and K be arbitrary two groups and let $\theta : K \to Aut(H)$ be a homomorphism. Then $H \rtimes_\theta K$ is the set of all ordered pairs $(h, k) \in H \times K$ under the binary operation*

$$(h, k)(h', k') = (h\theta_k(h'), kk')$$

In this definition, if we put $G = H \rtimes_\theta K$ then G actually forms a group under operation, and it is a semidirect product of H by K. Conversely, if G is a semidirect product of H by K, then G is isomorphic to $H \rtimes_\theta K$ for some $\theta : K \to Aut(H)$, hence $\theta_k : H \to H$ is given to be the automorphism defined by $\theta_k(h) = khk^{-1}$ for all $h \in H$ and $k \in K$ and the map defined by $(h, k) \to hk$ is an automorphism between $H \rtimes_\theta K$ and G.

Next, we show that residually finite property is preserved under a semidirect product.

LEMMA 2.2. *If H and K are residually finite groups then a semidirect product of H by K is also residually finite.*

Proof. Suppose that two groups H and K are residually finite and $H \rtimes_\theta K$ is a their semidirect product for some homomorphism $\theta : K \to Aut(H)$. For each non-identity element $(h, k) \in H \rtimes_\theta K$, we must construct a finite group G and a homomorphism $\pi : H \rtimes_\theta K \to G$ such that $\pi(h, k) \neq e_G$. Let (h, k) be a nonidentity element of $H \rtimes_\theta K$. Then both h and k are not identity elements simultaneously.

First, consider the case that $h \neq e_H$ and $k \neq e_K$. Since H and K are residually finite, there exist two finite groups A and B, and two homomorphisms $\phi : H \to A$ and $\psi : K \to B$ such that $\phi(h) \neq e_A$ and $\psi(k) \neq e_B$. Hence $\phi(H)$ and $\psi(K)$ are finite subgroups of A and B, respectively, such that $e_{\phi(H)} = e_A$ and $e_{\psi(K)} = e_B$. To make a Cartesian product set $\phi(H) \times \psi(K)$ a desired finite group, define a binary operation $*$ on $\phi(H) \times \psi(K)$ as follows;

$$(\phi(h), \psi(k)) * (\phi(h'), \psi(k')) = (\phi(h\theta_k(h')), \psi(kk'))$$

for all (h,k), $(h',k') \in H \times K$. Then it can be checked that $\phi(H) \times \psi(K)$ is a finite group under the operation $*$ such that the identity element is $(\phi(e_H), \psi(e_K))$ and the inverse element of $(\phi(h), \psi(k))$ is $(\phi(\theta_{k^{-1}}(h^{-1})), \psi(k^{-1}))$.

Let $\pi : H \rtimes_\theta K \to \phi(H) \times \psi(K)$ be the map defined by $\pi((h,k)) = (\phi(h), \psi(k))$ for each $(h,k) \in H \rtimes_\theta K$. Then

$$\begin{aligned}\pi((h,k)(h',k')) &= \pi(h\theta_k(h'), kk') \\ &= (\phi(h\theta_k(h')), \psi(kk')) \\ &= (\phi(h), \psi(k)) * (\phi(h'), \psi(k')) \\ &= \pi(h,k) * \pi(h',k').\end{aligned}$$

Thus π is a desired homomorphism sending (h,k) to a nonidentity element of a finite group $(\phi(H) \times \psi(K), *)$.

Next, let us consider the case that either of h and k is an identity element. If $h = e_H$, there exist a finite group B and a homomorphism $\psi : K \to B$ such that $\psi(k) \neq e_B$. Then the composite function $(\psi \circ pr_2) : H \rtimes_\theta K \to K \to B$ is a homomorphism with $(\psi \circ pr_2)(h,k) = \psi(k) \neq e_B$, where $pr_2 : H \tilde{\times} K \to K$ is the second projection.

If it is the case that $k = e_K$, let ϕ be a map from H to a finite group A with $\phi(h) \neq e_A$. Then the composition $\phi \circ pr_1 : H \rtimes_\theta K \to H \to A$ is a desirable homomorphism. Therefore the semidirect product $H \rtimes_\theta K$ is residually finite.

In a semidirect product $H \rtimes_\theta K$, if θ is the trivial automorphism of H, it is a direct product of H and K.

COROLLARY 2.3. *A direct product of residually finite groups is residually finite.*

3. Fiber bundles which are codimension 2 fibrators.

LEMMA 3.1([9]). *Finitely generated, residually finite groups and free groups are Hopfian groups.*

It is known that fundamental groups of 2-manifolds are residually finite [8], so that every closed 2-manifold has a Hopfian fundamental group.

If N is a Hopfian manifold and $p : M \to B$ is a proper map from an arbitrary $(n+k)$-manifold M to a finite dimensional space B such that each point preimage $p^{-1}(b)$ is homotopic equivalent to N, $p : M \to B$ is an approximate fibration over its continuity set $C \subset B$. For the case $k = 2$, $B \setminus C$ is locally finite and thus we can localize to the situation which B is an open disk and p is an approximate fibration over $B \setminus b$ for some $b \in B$. Furthermore, the Hopfian property of $\pi_1(N)$ and nonzero Euler characteristic of N force to extend the continuity set to the whole set B.

LEMMA 3.2[4]. *Every closed, Hopfian manifold with a Hopfian fundamental group and nonzero Euler characteristic is a codimension 2 fibrator.*

Since an aspherical closed manifold N having $\pi_1(N)$ Hopfian is a Hopfian manifold, every closed 2-manifold with negative Euler characteristic number is a codimension 2 fibrator

THEOREM 3.3. *Let N_1 and N_2 be aspherical closed manifolds with nonzero Euler characteristics. If both N_1 and N_2 have finitely generated free fundamental groups then their bundle structure $N_1 \tilde{\times} N_2$ is a codimension 2 fibrator.*

Proof. Let $p : N_1 \tilde{\times} N_2 \to N_1$ be the bundle projection. Since the base space N_1 is a compact manifold, p is a fibration and so $\chi(N_1 \tilde{\times} N_2) = \chi(N_1)\chi(N_2)$ ([14]). Consider the homotopy exact sequence between three objects;

$$\cdots \to \pi_n(N_2) \to \pi_n(N_1 \tilde{\times} N_2) \to \pi_n(N_1) \to \pi_{n-1}(N_2) \to \cdots$$

By the fact that N_1 and N_2 are aspherical, above exact sequence can be reduced to the short exact sequence as follows;

$$1 \to \pi_1(N_2) \to \pi_1(N_1 \tilde{\times} N_2) \to \pi_1(N_1) \to 1$$

Since $\pi_1(N_1)$ is a free group, this short exact sequence splits and thus $\pi_1(N_1 \tilde{\times} N_2)$ can be represented to be a semidirect product of $\pi_1(N_2)$ by $\pi_1(N_1)$. Moreover, free group is residually finite and Lemma 2.2 makes sure that $\pi_1(N_1 \tilde{\times} N_2)$ is residually finite. Of course, a semidirect product of finitely generated groups is finitely generated and so $N_1 \tilde{\times} N_2$

has the Hopfian fundamental group. Thus the aspherical manifold $N_1 \tilde{\times} N_2$ is a Hopfian manifold. By Lemma 3.2, $N_1 \tilde{\times} N_2$ is a codimension 2 fibrator.

By the classification theorem for compact surfaces, any compact orientable surface is homeomorphic to a sphere, or a connected sum of tori, so that it can be completely characterized by its Euler characteristic. In [3], R.J. Daverman claimed that any closed manifold that cyclically covers itself fails to be a codimension 2 fibrator. Since the only covering space of torus is itself, torus is not a codimension 2 fibrator, that is, the closed surface with zero Euler characteristic is not. On the other side, a closed surface with negative Euler characteristic is aspherical and has a free fundamental group, and finite product of such surfaces has the same attributes.

COROLLARY 3.4. *If N_1 and N_2 are finite product spaces of closed surfaces with negative Euler characteristics then the fibre bundle $N_1 \tilde{\times} N_2$ is a codimension 2 fibrator.*

Sometimes, the Hopfian property of fundamental group of a closed manifold makes N a Hopfian manifold. For low dimensional manifold, J.C. Hausmann proved the following useful result;

LEMMA 3.5 ([7]). *A closed, orientable n-manifold N is a Hopfian manifold provided $n \leq 4$ and $\pi_1(N)$ is Hopfian.*

THEOREM 3.6. *Let N be a closed surface with nonzero Euler characteristics. Then the fiber bundle $N \tilde{\times} S^2$ having 2-sphere S^2 as a fiber is a codimension 2 fibrator.*

Proof. We will consider two cases relative to Euler characteristics separately.

Case 1. $\chi(N) > 0$

Without loss of generality, we can assume that N is a 2-sphere. Since 2-sphere is a codimension 2 fibrator, the bundle projection is an approximate fibration and there is a exact sequence relating the homotopy groups of S^2, $N \tilde{\times} S^2$ and N, which informs that $N \tilde{\times} S^2$ is simply connected. Thus a simply connected space $N \tilde{\times} S^2$ is a codimension 2 fibrator.

Case 2. $\chi(N) < 0$

Consider the homotopy exact sequence;

$$1 \cong \pi_1(S^2) \to \pi_1(N\tilde{\times}S^2) \to \pi_1(N) \to 1$$

Since $\pi_1(N\tilde{\times}S^2)$ is isomorphic to $\pi_1(N)$, it is a Hopfian group. Hence $N\tilde{\times}S^2$ is a 4-manifold, so that we can apply Lemma 3.5 and thus it is a Hopfian manifold. As described in the proof of Theorem 3.3, the Euler characteristic of $N_1\tilde{\times}N_2$ is the same as the multiplication of $\chi(N_1)$ and $\chi(N_2)$. Therefore Lemma 3.2 guarantees that $N\tilde{\times}S^2$ is a codimension 2 fibrator.

COROLLARY 3.7. *Every product space $N_1 \times N_2$ of closed surfaces with nonzero Euler characteristics is a codimension 2 fibrator.*

Proof. By Corollary 3.4 and Theorem 3.6, it suffices to prove the case that $\chi(N_1) > 0$ and $\chi(N_2) < 0$. Since $\pi_1(N_1 \times N_2) \cong \pi_1(N_1) \oplus \pi_1(N_2) \cong \pi_2(N_2)$, it is a Hopfian group. Lemma 3.5 and Lemma 3.2 extract the desired result.

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