Pusan Kyŏngnam Math. J 11(1995), No. 2, pp 259–269

FORMAL MANIPULATIONS OF DOUBLE SERIES AND THEIR APPLICATIONS

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1.Introduction and Formulas

Formulas given here are in particular useful in the study of generating functions of sets of polynomials which are appeared in the theory of special functions. Of course they can be used in dealing with elementary functions. So we aim at giving some formulas for the rearrangement of double series without thinking of the convergence of them, some of which are already known or probably new. For such series the identities of this paper may be considered purely formal, but we shall find that the manipulative techniques are fully as useful as when applied to convergent series. We also show how some of them can be applied to deal with double series and elementary functions.

First we introduce five known formulas (see Rainville [3, pp. 56-58]): Note that $A_{x,y}$ denotes the function of two variables x and y throughout this paper.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{k,n-k}. \tag{1}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n+k}. \tag{2}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A_{k,n-2k}, \tag{3}$$

Received October 19, 1995.

where $[x]$ denotes the greatest integer in x.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n+2k}. \tag{4}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A_{k,n-k}. \tag{5}$$

Now we can easily generalize the formula (3) as follows:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} A_{k,n-pk}, \tag{6}$$

where p is any positive integer.

Indeed, consider the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n} t^{n+pk}$$

and in it collect powers of t, introducing new indices by

$$k = j, \qquad n + pj = m,$$

so that $n + pk = m$. Since $n \geq 0$, $k \geq 0$, we conclude that $m - pj \geq 0$, $j \geq 0$, from which $0 \leq pj \leq m$ and $m \geq 0$. Since $0 \leq j \leq m/p$ and j is an integer, the index j runs from 0 to the greatest integer in m/p. Thus we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n} t^{n+pk} = \sum_{m=0}^{\infty}\sum_{j=0}^{[m/p]} A_{j,m-pk} t^{m}$$

from which (6) follows by placing $t = 1$ and making the proper change of letters for the dummy indices on the right of the equation just above.

If the equation (6) is written in reverse order, we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n+pk}, \tag{7}$$

where p is any positive integer.

We give the following identity:

$$\sum_{\alpha=0}^{n}\sum_{k=\alpha}^{2\alpha} A_{\alpha,k} = \sum_{k=0}^{2n}\sum_{\alpha=\left[\frac{k+1}{2}\right]}^{k} A_{\alpha,k}. \tag{8}$$

Expanding the first member of (8), we have

$$S = \sum_{k=0}^{0} A_{0,k} + \sum_{k=1}^{2} A_{1,k} + \sum_{k=2}^{4} A_{2,k} + \cdots + \sum_{k=n}^{2n} A_{n,k}.$$

Writing the terms with equal indices of k in columns and adding these columns gives the desired result (8).

The use of the similar method to get (8) yields

$$\sum_{\alpha=0}^{n}\sum_{k=\alpha}^{3\alpha} A_{\alpha,k} = \sum_{k=0}^{3n}\sum_{\alpha=\left[\frac{k+2}{3}\right]}^{k} A_{\alpha,k}. \tag{9}$$

In view of (8) and (9), we can easily find the generalized formulas of (8) and (9):

$$\sum_{\alpha=0}^{n}\sum_{k=\alpha}^{p\alpha} A_{\alpha,k} = \sum_{k=0}^{pn}\sum_{\alpha=\left[\frac{k+p-1}{p}\right]}^{k} A_{\alpha,k}, \tag{10}$$

where p is any positive integer.

Similarly as in getting (10), we obtain

$$\sum_{n=0}^{\infty}\sum_{k=n}^{pn} A_{n,k} = \sum_{k=0}^{\infty}\sum_{n=\left[\frac{k+p-1}{p}\right]}^{k} A_{n,k}, \tag{11}$$

where p is any positive integer.

Setting $p = 2$ and $p = 3$ in (11) yields

$$\sum_{n=0}^{\infty}\sum_{k=n}^{2n} A_{n,k} = \sum_{k=0}^{\infty}\sum_{n=\left[\frac{k+1}{2}\right]}^{k} A_{n,k} \tag{12}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=n}^{3n} A_{n,k} = \sum_{k=0}^{\infty}\sum_{n=\left[\frac{k+2}{3}\right]}^{k} A_{n,k}. \tag{13}$$

We observe the following formula:

$$\sum_{k=\alpha}^{\infty}\sum_{n=k+\alpha}^{\infty} A_{k,n} = \sum_{n=2\alpha}^{\infty}\sum_{k=\alpha}^{n-\alpha} A_{k,n}. \tag{14}$$

Expanding the first member of (14), we have

$$\begin{aligned} S &= \sum_{n=2\alpha}^{\infty} A_{\alpha,n} + \sum_{n=2\alpha+1}^{\infty} A_{\alpha+1,n} + \sum_{n=2\alpha+2}^{\infty} A_{\alpha+2,n} + \cdots \\ &= \sum_{\beta=0}^{\infty} A_{\alpha,2\alpha+\beta} + \sum_{\beta=0}^{\infty} A_{\alpha+1,2\alpha+1+\beta} + \sum_{\beta=0}^{\infty} A_{\alpha+2,2\alpha+2+\beta} + \cdots . \end{aligned}$$

Adding S in columns gives

$$S = \sum_{\beta=0}^{0} A_{\alpha+\beta,2\alpha} + \sum_{\beta=0}^{1} A_{\alpha+\beta,2\alpha+1} + \sum_{\beta=0}^{2} A_{\alpha+\beta,2\alpha+2} + \cdots = \sum_{k=2\alpha}^{\infty}\sum_{n=\alpha}^{k-\alpha} A_{n,k},$$

from which (14) follows by interchanging n and k into k and n, respectively.

We find the following principle:

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{k,n}. \tag{15}$$

Expanding the first member of (15) gives

$$S=\sum_{n=0}^{\infty}A_{0,n}+\sum_{n=1}^{\infty}A_{1,n}+\sum_{n=2}^{\infty}A_{2,n}+\cdots .$$

Writing the terms with equal indices of n in columns and adding these columns, we obtain

$$S=\sum_{k=0}^{0}A_{k,0}+\sum_{k=0}^{1}A_{k,1}+\sum_{k=0}^{2}A_{k,2}+\cdots=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A_{k,n}.$$

Similarly as in getting previous 15 formulas, we write here four more principles:

(16) $$\sum_{k=1}^{\infty}\sum_{n=k}^{2k}A_{k,n}=\sum_{n=1}^{\infty}\sum_{k=0}^{[n/2]}A_{n-k,n}.$$

(17) $$\sum_{\alpha=1}^{k}\sum_{\beta=1}^{\alpha}A_{\alpha,\beta}=\sum_{\alpha=1}^{k}\sum_{\alpha=\beta}^{k}A_{\alpha,\beta}.$$

(18) $$\sum_{k=1}^{n}\sum_{m=1}^{n}A_{k,m}=\sum_{k=2}^{2n}\sum_{m=1}^{k-1}A_{k-m,m}.$$

(19) $$\sum_{n=1}^{\infty}\sum_{k=2}^{2n}A_{n,k}=\sum_{k=2}^{\infty}\sum_{n=\left[\frac{k+1}{2}\right]}^{\infty}A_{n,k}.$$

It should be remarked in passing that there is no bound to the number of such identities, whenever we need we can obtain our desired principles.

2. Applications

Consider the infinite series (Rainville [3, pp. 45-72]; see also Srivastava et al. [4, pp. 13-21]):

(20)
$$F(a,b;\ c;\ z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

$$(|z|<1;\ \ \mathrm{Re}(c-a-b)>0 \ \text{if}\ z=1;\ \ \mathrm{Re}(c-a-b)>-1 \ \text{if}\ z=-1)$$

which, for $a=c$ and $b=1$ (or, alternatively, for $a=1$ and $b=c$), reduces immediately to the relatively more familiar geometric series, and $(\alpha)_n$ denotes the Pochhammer symbol (or the *generalized factorial*, since $(1)_n=n!$) defined by

(21)
$$(\alpha)_0=1 \qquad \text{and} \qquad (\alpha)_n=\alpha(\alpha+1)\dots(\alpha+n-1)\ \ (n=1,2,3,\dots).$$

Hence (18) is called the hypergeometric series, or more precisely, Gauss's hypergeometric series after the famous German mathematician Carl Friedrich Gauss (1777-1855) who in the year 1812 introduced this series into analysis and gave the F-notation for it. It also denotes

$$F(a,b;\ c;\ z)={}_2F_1(a,b;\ c;\ z)$$
$$={}_2F_1\left[\begin{matrix}a, & b;\\ & c;\end{matrix}\right].$$

The enormous success of the theory of hypergeometric series in a single variable has stimulated the development of a corresponding theory in two and more variables. We present a brief account of Appell series.

Consider the product of two Gaussian series

$${}_2F_1(a,b;\ c;\ x){}_2F_1(a,b;\ c;\ y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!}.$$

This double series, in itself, yields nothing new, but by replacing one or more of the three pairs of products

$$(a)_m(a')_n, \qquad (b)_m(b')_n, \qquad (c)_m(c')_n,$$

by the corresponding expressions

$$(a)_{m+n}, \qquad (b)_{m+n}, \qquad (c)_{m+n},$$

we are led to five distinct possibilities of getting new double series. One such possibility, however, gives us the double series

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(a)_m(b)_m}{(c)_m} \frac{x^{m-n}}{(m-n)!} \frac{y^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} \frac{1}{m!} \sum_{n=0}^{m} \binom{m}{n} x^{m-n} y^n \\
&= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} \frac{(x+y)^m}{m!} = {}_2F_1(a, b; c; x+y),
\end{aligned}$$

where we use the principle (1) for the first equality.

The remaining four possibilities lead us to the four double hypergeometric series (known as *Appell series*), which are defined below (see Appell [1, p. 296, Equations (1)]:

$$\begin{aligned}
&F_1[a, b, b'; c; x, y] \\
&= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1\left[\begin{matrix} a+m, & b'; \\ & c+m; \end{matrix}\, y\right] \frac{x^m}{m!}, \\
&\max\{|x|, |y|\} < 1;
\end{aligned} \tag{22}$$

$$\begin{aligned}
&F_2[a, b, b'; c, c'; x, y] \\
&= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m}{m!} \frac{y^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1\left[\begin{matrix} a+m, & b'; \\ & c'; \end{matrix}\, y\right] \frac{x^m}{m!}, \\
&|x| + |y| < 1;
\end{aligned} \tag{23}$$

$$
(24)\qquad
\begin{aligned}
&F_3[a, a', b, b'; c; x, y] \\
&= \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1 \left[\begin{matrix} a', & b'; \\ & c+m; \end{matrix} \, y \right] \frac{x^m}{m!}, \\
&\max\{|x|, |y|\} < 1;
\end{aligned}
$$

$$
(25)\qquad
\begin{aligned}
&F_4[a, b; c, c'; x, y] \\
&= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{x^m}{m!} \frac{y^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1 \left[\begin{matrix} a+m, & b+m; \\ & c'; \end{matrix} \, y \right] \frac{x^m}{m!}, \\
&\sqrt{|x|} + \sqrt{|y|} < 1;
\end{aligned}
$$

here, as usual, the denominator parameters c and c' are neither zero nor a negative integer.

For further applications of principles (1) through (5), see Rainville [3, pp. 58-72] and see also Choi et al. [2].

Now we show how the principles introduced in section 1 can be applied to expand some elementary functions.

Let us expand $y = \log(a_0 + a_1 x + a_2 x^2)$ in powers of x. We may write

$$
y = \log a_0 + \log\left(1 + \frac{a_1}{a_0} x + \frac{a_2}{a_0} x^2\right).
$$

Then

$$
\begin{aligned}
y &= \log a_0 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{a_1}{a_0} x + \frac{a_2}{a_0} x^2\right)^k \\
&= \log a_0 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{\alpha=0}^{k} \binom{k}{\alpha} \left(\frac{a_1}{a_0}\right)^{k-\alpha} \left(\frac{a_2}{a_0}\right)^{\alpha} x^{k+\alpha}.
\end{aligned}
$$

Letting $k+\alpha=n$, then

$$y=\log a_0-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sum_{n=k}^{2k}\binom{k}{n-k}\left(\frac{a_1}{a_0}\right)^{2k-n}\left(\frac{a_2}{a_0}\right)^{n-k}x^n.$$

An application of the principle (16) to the equation just obtained yields
(26)

$$\begin{aligned} y&=\log a_0-\sum_{k=1}^{\infty}(-1)^n x^n\sum_{k=0}^{[n/2]}\frac{(-1)^k}{n-k}\binom{n-k}{k}\left(\frac{a_1}{a_0}\right)^{n-2k}\left(\frac{a_2}{a_0}\right)^k\\ &=\log a_0-\sum_{n=1}^{\infty}(-1)^n\left(\frac{a_1}{a_0}\right)^n x^n\sum_{k=0}^{[n/2]}\frac{(-1)^k}{n-k}\binom{n-k}{k}\left(\frac{a_0a_2}{a_1{}^2}\right)^k.\end{aligned}$$

In particular, setting $a_0=a_2=1$ and $a_1=-1$ in (26), we obtain

$$\log\left(1-x+x^2\right)=-\sum_{n=1}^{\infty}x^n\sum_{k=0}^{[n/2]}\frac{(-1)^k}{n-k}\binom{n-k}{k}. \tag{27}$$

If we replace x by 1 in (27), we get an interesting identity involving binomial coefficients:

$$\sum_{n=1}^{\infty}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{n-k}\binom{n-k}{k}=0. \tag{28}$$

Now

$$\log(1-x+x^2)=\log\frac{1+x^3}{1+x}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{3n}}{n}-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n},$$

then

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{3n}}{3n}-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{3n-1}}{3n-1}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{3n-2}}{3n-2}$$

therefore we have

$$\log(1-x+x^2) = -\sum_{n=1}^{\infty}(-1)^n\left[\frac{2x^{3n}}{3n}+\frac{x^{3n-1}}{3n-1}-\frac{x^{3n-2}}{3n-2}\right], \tag{29}$$

and we obtain

$$\sum_{k=0}^{\left[\frac{3m}{2}\right]}\frac{(-1)^k}{3m-k}\binom{3m-k}{k} = (-1)^m\frac{2}{3m}, \tag{30}$$

$$\sum_{k=0}^{\left[\frac{3m-1}{2}\right]}\frac{(-1)^k}{3m-1-k}\binom{3m-1-k}{k} = (-1)^m\frac{1}{3m-1}, \tag{31}$$

$$\sum_{k=0}^{\left[\frac{3m-2}{2}\right]}\frac{(-1)^k}{3m-2-k}\binom{3m-2-k}{k} = (-1)^{m-1}\frac{1}{3m-2}. \tag{32}$$

We conclude this paper by remarking that the principles of section 1 are no bound and so are their applications.

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