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ON RIGIDITY FOR REAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

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1. Introduction

Let $P_n(\mathbb{C})$ be an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4c and M be a $(2n-1)$ dimensional Riemannian manifold. Let ι be an isometric immersion of M into $P_n(\mathbb{C})$. An *almost contact structure* on M induced from the complex strcture $\widetilde{J}$ of $P_n(\mathbb{C})$ by ι will be denoted by (ϕ, ξ).

The problem with respect to the rigidity for real hypersurfaces in $P_n(\mathbb{C})$ has been studied by many geometers ([1], [2], [5] and [6] etc.). R.Takagi [6] proved that two isometric immersions of M into $P_n(\mathbb{C})$ are rigid if their second fundamental forms coincide. In [5], the same author and Y.J.Suh also obtaind the same conclusion if two isometric immersions have a principal direction in common and type number is not equal to 2 at each point of M, where the *type number* is defined as the rank of the second fundamental form. Recently, the author of the present paper together with Y.-W. Choe, I.-B. Kim and R.Takagi [2] showed that if there exists an m-dimensional subspace of the tangent space which is invariant under the actions of the shape operators, and the type number is not equal to 2 at each point of M, then two isometric immersions are rigid.

In this paper we shall prove the following.

MAIN THEOREM. *Let M be a $(2n-1)$-dimensional homogeneous Riemannian manifold, and ι and $\hat{\iota}$ be two isometric immersions of M into $P_n(\mathbb{C})$ $(n \geq 3)$. If there exists an m-dimensional subspace V of the tangent space at each point of M such that V is invariant under the actions of the shape operators of (M, ι) and $(M, \hat{\iota})$ $(2 \leq m \leq n-1)$,*

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then ι and $\hat{\iota}$ are rigid, that is, there exists an isometry φ of $P_n(\mathbb{C})$ such that $\varphi \circ \iota = \hat{\iota}$.

2. Preliminaries

We denote by $P_n(\mathbb{C})$ a complex projective space with the metric of constant holomorphic sectional curvature $4c$ and M a $(2n-1)$-dimensional Riemannian manifold. Let ι be an isometric immersion of M into $P_n(\mathbb{C})$. In the sequel the indices $i, j, k, l, \cdots$ run over the range $1, 2, \cdots, 2n-1$ unless otherwise stated. For a local orthonormal frame field $\{e_1, \cdots, e_{2n-1}\}$ of M, we denote its dual 1-forms by θ_i. Then the connection forms θ_{ij} and the curvature forms Θ_{ij} of M are defined by

$$d\theta_i + \sum \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0, \tag{1.1}$$

$$\Theta_{ij} = d\theta_{ij} + \sum \theta_{ik} \wedge \theta_{kj} \tag{1.2}$$

respectively. We denote the components of the shape operator or the second fundamental tensor A of (M, ι) by A_{ij}, and put $\psi_i = \sum A_{ij}\theta_j$. Then we have the equations of Gauss and Codazzi

$$\Theta_{ij} = \psi_i \wedge \psi_j + c\theta_i \wedge \theta_j + c\sum(\phi_{ik}\phi_{jl} + \phi_{ij}\phi_{kl})\theta_k \wedge \theta_l, \tag{1.3}$$

$$d\psi_i + \sum \psi_j \wedge \theta_{ji} = c\sum(\xi_j\phi_{ik} + \xi_i\phi_{jk})\theta_j \wedge \theta_k \tag{1.4}$$

respectively, where (ϕ_{ij}, ξ_k) is the almost contact structure on M. The tensor fields $A = (A_{ij}), \phi = (\phi_{ij})$ and $\xi = (\xi_i)$ on M satisfy

$$A_{ij} = A_{ji}, \tag{1.5}$$

$$\sum \phi_{ik}\phi_{kj} = \xi_i\xi_j - \delta_{ij}, \quad \sum \xi_j\phi_{ji} = 0, \quad \sum \xi_i{}^2 = 1, \tag{1.6}$$

$$d\phi_{ij} = \sum(\phi_{ik}\theta_{kj} - \phi_{jk}\theta_{ki}) - \xi_i\psi_j + \xi_j\psi_i, \tag{1.7}$$

$$d\xi_i = \sum(\xi_j\theta_{ji} - \phi_{ji}\psi_j). \tag{1.8}$$

For another isometric immersion $\hat{\iota}$ of M into $P_n(\mathbb{C})$, we shall denote the differential forms and tensor fields of $(M, \hat{\iota})$ by the same symbol as ones in (M, ι) but with a hat.

3. Proof of Main Theorem

Let ι and $\hat{\iota}$ be two isometric immersions of a $(2n-1)$-dimensional Riemannian manifold M into a complex projective space $P_n(\mathbb{C})$ ($n \geq 3$). In the following we assume that there exists an m-dimensional subspace V of the tangent space $T_p(M)$ of M at $p \in M$ such that V is invariant under the actions of the shape operators A of (M, ι) and $\hat{A}$ of $(M, \hat{\iota})$. If $m = 1$, then we have a principal direction in common and this case was studied in [5]. Since V is invariant under A and $\hat{A}$, so is the orthogonal complement $V^{\perp}$ of V. Therefore the case of $m \geq n$ can be alternated to that of $m \leq n-1$, and we have only to consider the case where $2 \leq m \leq n-1$.

LEMMA 2.1([2]). *If there is a subspace V mentioned above, then we have $\phi = \pm\hat{\phi}$.*

Proof of Main Theorem. Owing to Lemma 2.1 and $\Theta_{ij} = \hat{\Theta}_{ij}$, it follows from (1.3) that

$$\psi_i \wedge \psi_j = \hat{\psi}_i \wedge \hat{\psi}_j.$$

Then, by a well-known lemma of E.Cartan [1], we have at each point of M,

(2.1) if $t \geq 3$ or $\hat{t} \geq 3$, then $\psi_i = \varepsilon\hat{\psi}_i (\varepsilon = \pm 1)$ for $i = 1, \ldots, 2n-1$,

$$t + \hat{t} = 1 \ \text{ or } \ t = \hat{t}, \tag{2.2}$$

where t (resp. $\hat{t}$) denotes the type number of (M, ι) (resp. $(M, \hat{\iota})$). Since M is complete, it follows from a theorem in [4] due to the author of the present paper and R.Takagi that there exists a point p_0 such that $t(p_0) \geq n$. Let p be an arbitrary point of M. Then, since M is homogeneous, there exists an isometry g of M such that $g(p_0) = p$ and hence the type number is not smaller than n on M. Thus, we see from (2.1) that $A = \pm\hat{A}$ everywhere on M. Therefore ι and $\hat{\iota}$ are rigid (cf. Theorem 3.2 in [6]). □

COROLLARY 2.2. *Let M be a $(2n-1)$-dimensional homogeneous Riemannian manifold, and ι be an isometric immersion of M into $P_n(\mathbb{C})$ $(n \geq 3)$. Assume that there exists an m-dimensional subspace V of the tangent space at each point of M such that V is invariant under the actions of the shape operators of (M, ι) and $(M, \hat{\iota})$ $(2 \leq m \leq n-1)$. Then $\iota(M)$ is an orbit under an analytic subgroup of the projective unitary group $PU(n+1)$.*

Note that all real hypersurfaces in $P_n(\mathbb{C})$ obtained as orbits under analytic subgroups of the projective unitary group $PU(n+1)$ are completely classified in [6].

Proof of Corollary 2.2. For any isometry g of M we have another isometric immersion $\hat{\iota} = \iota \circ g$ of M into $P_n(\mathbb{C})$. Here we note that there exists a point p_0 on M such that $t(p_0) \geq n$. Let p be an arbitrary point of M. Then, since M is homogeneous, there exists an isometry g of M such that $g(p_0) = p$. By (2.2) we find

$$t(p_0) = \hat{t}(p_0) = t(p)$$

because $t(p_0) \geq 3$. Thus we have $t(p) \geq 3$.

Now by a theorem in [2], there exists an isometry φ_g of $P_n(\mathbb{C})$ such that $\varphi_g \circ \iota = \iota \circ g$, and $\iota(M)$ is just an orbit under the analytic subgroup $\{\varphi_g; g \in I(M)\}$ of $PU(n+1)$, where $I(M)$ denotes the group of all isometries of M. □

REMARK 2.3. The Main Theorem and Corollary 2.2 are not valid for a complex hyperbolic space $H_n(\mathbb{C})$ with negative constant holomorphic sectional curvature.

Open problem. Let M be an $(2n-1)$-dimensional Riemannian manifold, and ι and $\hat{\iota}$ be two isometric immersions of M into $P_n(\mathbb{C})$. Then are ι and $\hat{\iota}$ rigid ?

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