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INTERGRAL REPRESENTATION OF VECTOR–VALUED CONTINUOUS FUNCTIONS

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1. Introduction

Let S be a compact Hausdorff space, let X, Y be locally convex Hausdorff spaces over real and complex field. Let $C(S,X)$ denote the continuous functions from S into X with the topology of uniform convergence.

The purpose of this paper is to give an integral representation for continuous linear operator T on $C(S,X)$ into Y by means of integrals with respect to $L(X,Y)$ and we investigate some problems of the theory of vector-valued functions for an operator-valued measure.

2. Preliminaries and Notations

Let Σ be an σ-algebra of the closed subsets of S and $L(X,Y)$ be the space of all continuous linear operators on X into Y. Let Y' and Y'' be dual and bidual of Y, respectively.

For each continuous semi-norm q of Y there exists a continuous semi-norm p on X such that $\{q(T(x)); x \in B_p\}$ is bounded, where $B_p = \{x \in X; p(x) \leq 1\}$. By B_p^0 we mean the polar set of B_p, i.e. the set $x' \in X'$ with $| < x, x' > | \leq 1$ for all $x \in B_p$ and $p(x) = \sup\{| < x, x' > |; x' \in B_p^0\}$. The topology of $C(S,X)$ is generated by the seminorms $p(f) = \sup_{s \in S} p(f(s))$, and the topology for Y'' is generated by the seminorms $q''(y'') = \sup_{y' \in B_q^0} | < y', y'' > |$. If $E \in \Sigma$ we denote the characteristic function of E by χ_E.

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If $E \in \Sigma$ and $x \in X$ we identify the simple function $\chi_E \cdot x$ as an element of $C''(S,X)$ since this identification is an isometric isomorphism in [4] and [8]. The linear operator $T; C(S,X) \to Y$ is continuous if and only if there exists a pairing (p,q) such that $\|T\|_{(p,q)} = \sup\{q(T(f)); p(f) \leq 1\}$. It is well known that T''(the bitranspose of T) maps $C''(S,X)$ into Y'', and $\|T\|_{(p,q)} = \|T''\|_{(p,q)}$.

DEFINTION 2.1. *An operator-valued measure $\mu; \Sigma \to L(X,Y)$ said to be of bounded (p,q)-variation on $E \in \Sigma$ for a continuous semi-norm $p(q)$ on $X(Y)$ if*

$$\{q(\sum_{i=1}^{\infty} \mu(E_i)x_i); \quad E_i \cap E_j = \phi(i \neq j), x_i \in B_p\}$$

is bounded and we define the (p,q)-variation of μ on $E \in \Sigma$,

$$\| \mu \|_{(p,q)} = \sup_{y' \in B_q^\circ} \{q(\sum_{i=1}^{n} y' \mu(E_i)x_i); y' \in Y', x_i \in B_p\}.$$

DEFINITION 2.2. *A function $f; S \to X$ said to be μ- integrable with respect to an operator-valued measure if*

(1) *f is $y'\mu$-integrable (in the sense of [3]), and*
(2) *for each $E \in \Sigma$, there is an element $y_E \in Y$ such that*

$$y'(y_E) = \int_E f y' \mu(ds), \quad for \quad y' \in Y'.$$

Since Y is a locally convex Hausedorff space, we denote $y_E = \int_E f(s)\mu(ds)$ and y_E is unique whenever it exists. It is well known that X-valued simple function is μ-integrable and the integral of such a function is given by

$$\int_E f\mu(ds) = \sum_{i=1}^{n} \mu(E \cap E_i)x_i$$

is an Y-valued measure on Σ.

DEFINITION 2.3. *$L(X, Y'')$-valued measure, defined on Σ is said to be weakly regular if the set function $y'\mu(\cdot)x$ is regular for $x \in X$ and $y' \in Y'$.*

From the above definition we have that

$$\begin{aligned} < \mu(E)x, y' > &= < T''(\chi_E \cdot x), y' > \\ &= < \chi_E \cdot x, T'y' > \\ &= y'\mu(E)x \end{aligned}$$

which weakly operator valued regular mesure. Suppose μ is any weakly regular operator-measure of bound (p, q'') such that $T(f) = \int f du$, then

$$< T(f \cdot x), y' > = \int f y'\mu(ds) = < f \cdot x, T'y' > = < \mu(E), y' >, x \in X.$$

Since Y is locally convex space for $y' \in Y', f \in C(S, X)$, if

$$\int f y'\mu(ds) = \int f y'\lambda(ds),$$

then we have $y'\mu = y'\lambda$. Hence $\mu = \lambda$.

LEMMA 2.4. *[5] For $E_i \in \Sigma, x_i \in X, E_i \bigcap E_j = \phi (i \neq j)$, $i, j = 1, 2, ..., n$, we have that*

$$q''(\sum_{i=1}^{n} \chi_{E_i} \cdot x_i) \leq \max p(x_i).$$

For q'' on Y'' there exists a p such that T is (p, q)-related and so T'' is (p'', q'')-related such that

$$\begin{aligned} q''(\sum_{i=1}^{n} \mu(E_i)x_i) &= q''(T''(\sum_{i=1}^{n} \chi_{E_i} \cdot x_i)) \\ &\leq ||T||_{(p'', q'')} p''(\sum_{i=1}^{n} \chi_{E_i} \cdot x_i) \\ &\leq ||T||_{(p,q)} \max p(x_i). \end{aligned}$$

Therefore we see that $\mu(E) \in L(X, Y'')$, for each $E \in \Sigma$, since $q''(\mu(E)x) \leq ||T||_{(p,q)} p(x)$.

PROPOSITION 2.5. *Let $T; C(S,X) \to Y$ be a continuous linear operator, then the weakly operator-valued regular measure μ defined on Σ with values in $L(X,Y'')$, given by*

$$\mu(E)x = T''(\chi_E \cdot x) \quad for \quad E \in \Sigma, x \in X.$$

Proof. For $y' \in Y'$ and $x_i \in X \quad (i = 1, 2, ..., n)$,

$$\begin{aligned} q''(\sum_{i=1}^{n} \mu(E_i)x_i) &= \sup_{y' \in B_q^\circ}(y'T''(\sum_{i=1}^{n} \chi_{E_i} \cdot x_i)) \\ &\leq ||T||_{(p'',q'')} p''(\sum_{i=1}^{n} \chi_{E_i} \cdot x_i) \\ &\leq ||T||_{(p,q)} \max_i p(x_i). \end{aligned}$$

For $y' \in Y'$ and $x \in X$, let $\lambda(E) = y'\mu(E)x$, then

$$y'\mu(E)x = y'(T''(\chi_E \cdot x)) = (\chi_E \cdot x)(T'y') \quad for \quad E \in \Sigma,$$

which is regular measure.

3. Representation of continuous linear operator

Every $L(X,Y)$-valued measure μ on Σ may be considered as being $L(X,Y'')$-valued, by the canonical mapping of X into X''. Therefore we can define $< \mu(E)x, y' >= y'\mu(E)x$ and we have

$$q(y'\mu(E)x) \leq q(\mu(E))p(x), E \in \Sigma, x \in X.$$

Let $\mu; \Sigma \to L(X,Y'')$ be an operator-valued measure. By $M(\Sigma, X')$, the space of all regular X'-valued measures of finite variations on Σ, $\quad y'\mu \in M(\Sigma, X')$ is finitely additive.

THEOREM 3.1. *Let S be a locally convex Hausdorff space and $T; C(S,X) \to Y$ be continuous linear operator. Then there exists a unique operator-valued measure $\mu; \Sigma \to L(X,Y'')$ such that*

(1) *the linear map $y' \to y'\mu$ on Y' into $M(\Sigma, X')$ for each $y' \in Y'$ is continuous,*

(2) *if* T *is* (p,q)*-defined operator, then we have*

$$\| \mu \|_{(p,q)} = \| T \|_{(p,q)} .$$

(3) $y'(T(f)) = \int f y' \mu(ds)$, $f \in C(S,X)$, $y' \in Y'$
(4) $T'y' = y'\mu \quad$ *for* $\quad y' \in Y'$.

Conversely if $\mu; \Sigma \to L(X, Y'')$ *has properties (1) and (2), then the linear operator* $T; C(S,X) \to Y$ *defined by (3) is continuous and* (p,q)*-defined operator as (2), and whose adjoint is given by (4).*

Proof. For $E \in \Sigma$, $\mu(E); X \to Y''$ is linear and from Lemma 3.2 we have

$$\mu(E)x = T''(\chi_E \cdot x), \quad x \in X.$$

For each continuous seminorms p, q on X, Y, respectively, we define

$$y \in Y, \quad y \to q(y) = |< y, y' >| \quad and$$
$$\begin{aligned} q(y'\mu(E)x) &= q(y'T''(\chi_E \cdot x)) = q(T'y'(\chi_E \cdot x)) \\ &\le \sup_{\|f\| \le 1} q((T'y')(f \cdot x)) \\ &\le \|T\|_{(p,q)} p(f \cdot x) \le \|T\|_{(p,q)} \|f\|_{C(S,X)} \cdot p(x). \end{aligned}$$

Furthermore from (3) we have the following property

$$\begin{aligned} y'(T(f)) &= y'(T(\sum_{i=1}^{n} \chi_{E_i} \cdot x_i)) \\ &= y'(\sum_{i=1}^{n} \mu(E_i)x_i) = \sum_{i=1}^{n} y'\mu(E_i)x_i. \end{aligned}$$

Thus it follows that

$$y'(T(f)) = \int f y' \mu(ds) \quad for \quad f \in C(S,X)$$

which complete the proof of (3). Let us prove relation (2).

$$\begin{aligned} \|T\|_{(p,q)} &= \sup\{q(T(f)); p(f) \le 1\} \\ &= \sup_{p(f) \le 1} \sup_{y' \in B_q^o} (y'(T(f))) \\ &= \sup_{y' \in B_q^o} \sup_{p(f) \le 1} (y'(T(f))). \end{aligned}$$

On the other hand, we have

$$\sup_{y'\in B_q^\circ}\sup_{p(f)\le 1}(y'(T(f))) = \sup_{y'\in B_q^\circ}\sup|\sum_{i=1}^{n} y'\mu(E_i)x_i|$$

$$= \sup_{y'\in B_q^\circ}|y'(\sum_{i=1}^{n}\mu(E_i)x_i)| = \sup q(\sum_{i=1}^{n}\mu(E_i)x_i) = ||\mu||_{(p,q)},$$

where the supremum is taken over all Σ-partition of S into $E_i \in \Sigma$ and all possible collections $x_i \in X$ with $p(x_i) \le 1$, which proves (2) and (4). Conversely let $\mu;\Sigma \to L(X,Y'')$ satisfy (1) and (2), then for $f \in C(S,X), T(f) \in Y$, where T is defined by (3), the linear mapping $y' \to y'\mu$ of Y' into $M(\Sigma, X')$ is continuous with respect to the Y-topology in Y' and $C(S,X)$-topology in $C'(S,X)$. Thus the linear operator $T(f) = \int f\mu(ds)$ of $C(S,X)$ into Y is continuous and (2) holds.

COROLLARY 3.2. *Let Y be semi-reflexive and $T; C(S,X) \to Y$ be continuous linear operator. Then there exists a unique operator-valued measure $\mu;\Sigma \to L(X,Y)$ such that*

(1) *the mapping $y' \to y'\mu$ on Y' into $M(\Sigma, X')$ is continuous,*
(2) *if T is (p,q)-defined operator, then $\| \mu \|_{(p,q)} = \| T \|_{(p,q)}$.*
(3) $T(f) = \int f\mu(ds)$, $f \in C(S,X)$
(4) $T'y' = y'\mu$

Conversely, if $L(X,Y)$-valued measure which satisfies that (1), then the linear operator $T; C(S,X) \to Y$ is defined by (3) and (4) is continuous with the condition (2).

Proof. Since $Y'' = Y$, the proof can be obtained by a slight modification of the proof of the above theorem.

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