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ON THE BOUNDEDNESS OF HARDY-LITTLEWOOD MAXIMAL OPERATOR AND SAWYER'S CONDITION

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1. Introduction

Let f be a locally integrable function in R^n. For $x \in R^n$,

$$\mathcal{M}f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes Q containing x and $|Q|$ stands for the Lebesgue measure of Q. This operator $\mathcal{M}$ is called the Hardy-Littlewood maximal operator. By many authors such as C. Fefferman, E.M. Stein and E.T. Sawyer are studied this maximal operator ,singular integral and interpolation between function space e.t.c, $\cdots$([1],[2],[3],[6],[8]).In this many problem, there is boundedness of Hardy-Littlewood maximal operator i,e.,

Given p $(1 < p < \infty)$, determine those pairs of weights on R^n. (u, w), for which $\mathcal{M}$ is of strong type (p, p) with respect to the pair of measures $(u(x)dx, w(x)dx)$, that is, for which we have an inequality:

$$\left(\int_{R^n} (\mathcal{M}f(x))^p u(x) dx\right)^{\frac{1}{p}} \leq C\left(\int_{R^n} |f(x)|^p w(x) dx\right)^{\frac{1}{p}}.$$

In [5], This answer is provided by the S_p theory.

In this paper, we will prove this inequality by generalization $\mathcal{M}_s f$ instead of $\mathcal{M}f$.

2.Preliminaries

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For locally integrable function f in R^n and $x \in R^n$, we will denote $\mathcal{M}_s f$ as generalization of $\mathcal{M}f$. i,e. ,

$$\mathcal{M}_s f(x) = (\sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|^s dy)^{\frac{1}{s}}, \quad 1 \le s.$$

Also, given a cube Q_0 that is "right open", we choose a system of-coordinates of R^n with respect to Q_0 is $[0,1)^n$. A cube Q is then a dyadic cube with respect to Q_0 if it is of the form $Q = \{x \in R^n : x = (x_1, x_2, ..., x_n) | k_i 2^{-k} \le x_i < (k_i + 1)2^{-k}\}$, where the $k_i's$ and k range over integer Z. By a basis in R^n, we denote $\mathcal{D}$ by the collection of the open dyadic cubes. The properties of dyadic cube is in [4] and [5]. Dyadic maximal operator $\mathcal{N}f$ is Hardy-Littlewood maximal operator with dyadic cube.

Let dx denote Lebesgue measure on R^n. We denote by $\omega : R^n \to R$ a weight; that is, a positive, measurable, and locally integrable function. Also, we shall write $\omega(Q) = \int_Q \omega(x)dx$ when $Q \subset R^n$ is measurable. For $1 < p < \infty$, couple of weights (u, w) satisfies the *Sawyer's* condition (S_p condition) if $(\frac{\sigma(Q)}{|Q|})^p u(Q) = \int_Q |\mathcal{M}(\sigma\chi_Q)(x)|^p u(x)dx \le C\sigma(Q)$, where $\sigma = w^{\frac{-1}{p-1}}$. Similarly, we define $S_{ps}, (1 \le s)$ condition by $\int_Q |\mathcal{M}_s(\sigma^{\frac{1}{s}}\chi_Q)(x)|^{ps} u(x)dx \le C\sigma(Q)$. This S_p condition is stronger then A_p condition ([5],[9]). However, for $u = w$, we know that S_p is equivalent to A_p ([5]). This A_p condition is related to the weak type of maximal operator.

3. Results

LEMMA 1. *For every integer k, every locally integrable function f in R^n and $x \in R^n$:*

$$\mathcal{M}_s^{(2^k)} f(x) \le 2^{(\frac{2n+s+ns}{s})} \frac{1}{|Q(0.2^{k+2})|} \int_{Q(0,2^{k+2})} (\tau_{-t} \circ \mathcal{N}_s \circ \tau_t) f(x) dt,$$

where $\tau_t g(x) = g(x - t)$.

Proof. Since

$$(\tau_{-t} \circ \mathcal{N}_s \circ \tau_t)f(x) = \mathcal{N}_s(\tau_t f)(x+t)$$
$$= (\sup_{x+t\in Q\in\mathcal{D}} \frac{1}{|Q|}\int_Q |f(y-t)|^s dy)^{\frac{1}{s}}$$
$$= (\sup_{x\in Q-t, Q\in\mathcal{D}} \frac{1}{|Q|}\int_{Q-t} |f(z)|^s dz)^{\frac{1}{s}}.$$

Thus $\tau_{-t}\circ\mathcal{N}_s\circ\tau_t$ is simply the operator $\mathcal{M}_{\mathcal{D}-t,s}$ associated to the basis $\mathcal{D}-t$ formed by the cubes $Q-t$ with Q dyadic, that is, the translates by $-t$ of the dyadic cubes. Given k, f and x, by definition of $\mathcal{M}_s^{(2^k)}f(x)$, there will exist a cube R of side length $\leq 2^k$ such that $x \in R$ and

$$\frac{1}{2}\mathcal{M}_s^{(2^k)}f(x) \leq (\frac{1}{|R|}\int_R |f(y)|^s dy)^{\frac{1}{s}}.$$

Let j be an integer such that $2^{j-1} <$ side length of $R \leq 2^j$, where $j \leq k$. Consider the set Ω consisting of those $t \in Q(0, 2^{k+2})$ for which there is some $Q \in \mathcal{D} - t$ with side length equal to 2^{j+1} and such that $R \subset Q$.For every $t \in \Omega$,
we have;

$$\frac{1}{2}\mathcal{M}_s^{(2^k)}f(x) \leq (\frac{1}{|R|}\int_R |f(y)|^s dy)^{\frac{1}{s}}$$
$$\leq (\frac{2^{2n}}{|Q|}\int_Q |f(y)|^s dy)^{\frac{1}{s}}$$
$$\leq 2^{\frac{2n}{s}}(\tau_{-t}\circ\mathcal{N}_s\circ\tau_t)f(x).$$

Since geometrical observation that the measure of Ω is at least $2^{(k+2)n} = \frac{|Q(0,2^{k+2})|}{2^n}$. Then

$$\mathcal{M}_s^{(2^k)}f(x) \leq 2^{(\frac{2n+s}{s})}\frac{1}{|\Omega|}\int_\Omega (\tau_{-t}\circ\mathcal{N}_s\circ\tau_t)f(x)dt$$
$$\leq 2^{(\frac{2n+s+ns}{s})}\frac{1}{|Q(0,2^{k+2})|}\int_{Q(0,2^{k+2})} (\tau_{-t}\circ\mathcal{N}_s\circ\tau_t)f(x)dt.$$

THEOREM 2. *([7]) Let $1 \leq s < p < \infty$ and (u, w) be a couple of weights in R^n. Then, the following two conditions are equivalent: (a) $\mathcal{N}_s$ is bounded from $L^{ps}(w)$ to $L^{ps}(u)$ (b) There is a constant C such that, for every dyadic cube Q:*

$$\int_Q |\mathcal{N}_s(\sigma^{\frac{1}{s}}\chi_Q)(x)|^{ps} u(x)dx \leq C\sigma(Q).$$

THEOREM 3. *Let $1 \leq s < p < \infty$ and (u, w) be a couple of weights in R^n. Then, the following two conditions are equivalent: (a) $\mathcal{M}_s$ is bounded from $L^{ps}(w)$ to $L^{ps}(u)$ (b) $(u, w) \in S_{ps}$ condition for maximal operator $\mathcal{M}_s$:*

$$\int_Q |\mathcal{M}_s(\sigma^{\frac{1}{s}}\chi_Q)(x)|^{ps} u(x)dx \leq C\sigma(Q) < \infty, \quad \sigma = w^{\frac{-1}{p-1}}.$$

Proof. That (a) implies (b) is almost immediate. If (a) holds, we have an inequality

$$\int_{R^n} |\mathcal{M}_s f(x)|^{ps} u(x)dx \leq C \int_{R^n} |f(x)|^{ps} w(x)dx$$

valid for every function f. If we apply this inequa lity to the function $f = \sigma^{\frac{1}{s}}\chi_Q$, we obtain:

$$\begin{aligned}\int_{R^n} |\mathcal{M}_s(\sigma^{\frac{1}{s}}\chi_Q)|^{ps} u(x)dx &\leq C \int_Q \sigma(x)^p w(x)dx \\ &= C\sigma(Q).\end{aligned}$$

So weights (u, w) are in S_{ps} class for maximal operator $\mathcal{M}_s$. Now, we show that(b) implies (a). But, after Lemma 1, the boundedness of $\mathcal{M}_s$ is equivalent to the uniformly boundedness of operators $\tau_{-t} \circ \mathcal{N}_s \circ \tau_t$ for $t \in R^n$. Since

$$\int_{R^n} |(\tau_{-t} \circ \mathcal{N}_s \circ \tau_t)f(x)|^{ps} u(x)dx = \int_{R^n} |\mathcal{N}(\tau_t f)(y)|^{ps} u(y-t)dy$$

and

$$\int_{R^n} |f(x)|^{ps} w(x)dx = \int_{R^n} |\tau_t f(y)|^{ps} w(y-t)dy,$$

we see that the uniformly boundedness of the operator $\tau_{-t} \circ \mathcal{N}_s \circ \tau_t$ between $L^{ps}(w)$ to $L^{ps}(u)$ is equivalent to the fact that the couple $(\tau_t u, \tau_t w)$ satisfy condition (b) in theorem 2 with a constant independent of t. But this fact follows quite easily from our condition (b). Indeed, for every $t\ inR^n$ and any dyadic cube Q ,we have;

$$\begin{aligned}
\int_Q |\mathcal{N}_s((\tau_t \sigma^{\frac{1}{s}})\chi_Q)(x)|^{ps} (\tau_t u)(x)dx &= \int_Q |\mathcal{N}_s(\tau_t(\sigma^{\frac{1}{s}}\chi_{Q-t}))(x)|^{ps} u(x-t)dx \\
&= \int_{Q-t} |(\tau_{-t} \circ \mathcal{N}_s \circ \tau_t)(\sigma^{\frac{1}{s}}\chi_{Q-t})(y)|^{ps} u(\\
&\le \int_{Q-t} |\mathcal{M}_s(\sigma^{\frac{1}{s}}\chi_{Q-t})(y)|^{ps} u(y)dy \\
&\le C\sigma(Q-t) \\
&= C\int_{Q-t} \sigma(x)dx \\
&= C(\tau_t\sigma)(Q).
\end{aligned}$$

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