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DUAL OPERATOR ALGEBRAS AND PROPERTIES $D_{\theta,\gamma}$

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1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that cointains T and $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology. Moreover, let $Q_{\mathcal{A}_T}$ denote the quotient space $\mathcal{C}_1(\mathcal{H})/\perp_{\mathcal{A}_T}$, where $\mathcal{C}_1(\mathcal{H})$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and $\perp_{\mathcal{A}_T}$ denotes the preannihilator of $\mathcal{A}_T$ in $\mathcal{C}_1(\mathcal{H})$. For a brief notation, we shall denote $Q_{\mathcal{A}_T}$ by Q_T. One knows that $\mathcal{A}_T$ is the dual space of Q_T and that the duality is given by

$$(1) \qquad \langle A, [L]\rangle = \mathrm{tr}(AL), \quad A \in \mathcal{A}_T, [L] \in Q_T.$$

The Banach space Q_T is called a predual of $\mathcal{A}_T$. For x and y in $\mathcal{H}$, we can write $x \otimes y$ for the rank one operator in $\mathcal{C}_1(\mathcal{H})$ defined by

$$(2) \qquad (x \otimes y)(u) = (u, y)x \quad for \quad all \quad u \in \mathcal{H}.$$

The theory of dual algebras is applied to the study of invariant subspaces, dilation theory, and reflexivity. The class $\mathbb{A}_{m,n}$ were defined by Bercovici-Foias-Pearcy in [2]. Also these classes are closely related to the study of the theory of dual algebras. Especially, Apostol-Bercovici-Foias-Pearcy established property $X_{\theta,\gamma}$, and researched a relationship with the class $\mathbb{A}_{\aleph_0}$ in [1]. In this paper, we define a new property $D_{\theta,\gamma}$

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which is the generalization of property $X_{\theta,\gamma}$ and obtain some results related to dual algebras.

2. Preliminaries and properties $D_{\theta,\gamma}$

The notation and terminology employed herein agree with those in [3], [4] and [8].

DEFINITION 2.1. *Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and θ is a nonnegative real number. We denote by $\mathcal{D}_\theta(\mathcal{A})$ the set of all $[L]$ in $Q_{\mathcal{A}}$ such that there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ of vectors from $\mathcal{H}$ satisfying*

(a) $\limsup_{i\to\infty} \|[x_i \otimes y_i] - [L]\| \leq \theta$,
(b) $\|x_i\| \leq 1, \|y_i\| \leq 1, 1 \leq i < \infty$, *and*
(c) *sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ converge weakly to zero.*

For $0 \leq \theta < \gamma < \infty$, the dual algebra $\mathcal{A}$ is said to have property $D_{\theta,\gamma}$ if the closed absolutely convex hull of the set $\mathcal{D}_\theta(\mathcal{A})$ contains the closed ball $B_{0,\gamma}$ of radius γ centered at the origin in $Q_{\mathcal{A}}$:

$$\overline{aco}(\mathcal{D}_\theta(\mathcal{A})) \supset \{[L] \in Q_{\mathcal{A}} : \|[L]\| \leq \gamma\} = B_{0,\gamma}. \tag{3}$$

LEMMA 2.2 ([3], PROPOSITION 1.21). *Let X be a complex Banach space, let M be a positive number, and let E be a subset of X. Then*

$$\|\phi\| \leq M \sup_{x\in E} |\phi(x)|, \quad \phi \in X^* \tag{4}$$

if and only if $\overline{aco}(E)$ contains the closed ball of radius $1/M$ about the origin in X.

If $\mathcal{A}$ is a subalgebra of $\mathcal{L}(\mathcal{H})$, we denote by $\mathcal{M}_n(\mathcal{A})$ subalgebra of $\mathcal{L}(\mathcal{H}^{(n)})$ consisting of all $n \times n$ matrices with the entries from $\mathcal{A}$.

LEMMA 2.3 ([3], PROPOSITION 2.2). *If n is a positive integer and $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra, then $\mathcal{M}_n(\mathcal{A})$ is a dual subalgebra of $\mathcal{L}(\mathcal{H}^{(n)})$. The predual $Q_{\mathcal{M}_n(\mathcal{A})}$ can be identified, as a Banach space,* *with the Banach space $\mathcal{M}_n(Q_{\mathcal{A}})$ consisting of all $n \times n$ matrices with entries from $Q_{\mathcal{A}}$. Under this identification the duality between $M_n(\mathcal{A})$ and $M_n(Q_{\mathcal{A}})$ is given by*

$$(5) \qquad \langle (T_{ij}), ([L_{ij}]) \rangle = \sum_{i,j=1}^{n} \langle T_{ij}, [L_{ij}] \rangle,$$

$$\text{where} \quad (T_{ij}) \in \mathcal{M}_n(\mathcal{A}), \quad ([L_{ij}]) \in \mathcal{M}_n(Q_{\mathcal{A}}),$$

and the norm on $\mathcal{M}_n(Q_{\mathcal{A}})$ is the norm that accrues to it as a linear manifold in $\mathcal{M}_n(\mathcal{A})^*$. In particular, if $\tilde{x} = (x_1, \cdots, x_n)$ and $\tilde{y} = (y_1, \cdots, y_n)$ belong to $\mathcal{H}^{(n)}$, then $[\tilde{x} \otimes \tilde{y}]_{Q_{\mathcal{M}_n(\mathcal{A})}}$ is identified with the $n \times n$ matrix $([x_j \otimes y_j]_{Q_{\mathcal{A}}})$.

For a dual algebra $\mathcal{A}$, we will denote by $\mathcal{A}_1$ the set $\{A \in \mathcal{A} : \|A\| \leq 1\}$. For $x \in \mathcal{H}$, if we define the map $\rho_x : \mathcal{A} \to \mathcal{H}$ by $\rho_x(A) = Ax, A \in \mathcal{A}$, then $\mathcal{M}_c = \mathcal{M}_c(\mathcal{A})$ denote the set

$$\{x \in \mathcal{H} : \rho_x(\mathcal{A}_1) \quad \text{is} \quad \textit{norm} \quad \textit{compact}\}.$$

Of course $\mathcal{M}_c$ might be (0) or $\mathcal{H}$. This motivates the following definition.

DEFINITION 2.4 ([7], DEFINITION 3.3). *Let $\mathcal{A}$ be a dual algebra.*
(*a*) *$\mathcal{A} \in A_{0\cdot}(\mathcal{H})$ if $\mathcal{M}_c = \mathcal{H}$.*
(*b*) *$\mathcal{A} \in A_{1\cdot}(\mathcal{H})$ if $\mathcal{M}_c = (0)$.*
(*c*) *$\mathcal{A} \in A_{\cdot\alpha}(\mathcal{H})$ if $\mathcal{A}^* \in A_{\alpha\cdot}, \alpha = 0, 1$.*
(*d*) *$\mathcal{A} \in A_{\alpha\beta}(\mathcal{H})$ if $\mathcal{A} \in A_{\alpha\cdot} \cap A_{\cdot\beta}, \alpha = 0, 1, \beta = 0, 1$.*

We write $A_{\alpha\beta}$ for $A_{\alpha\beta}(\mathcal{H})$ when there is no possibility of confusion. The next result provides a link between $\mathcal{M}_c$ and $Q_{\mathcal{A}}$.

LEMMA 2.5 ([7], PROPOSITION 3.4). *Let $\mathcal{A}$ be a dual algebra and $x \in \mathcal{H}$. The following are equivalent.*
(*a*) *$x \in \mathcal{M}_c$.*

(b) $\rho_x : \mathcal{A} \to \mathcal{H}$ *is a compact operator.*
(c) If $\{y_n\}$ *is a sequence in* $\mathcal{H}$ *with* $y_n \to 0$ *weakly, then* $\|[x \otimes y_n]\| \to 0$.

3. Dual operator algebras and properties $\mathcal{D}_{\theta,\gamma}$

LEMMA 3.1. $[L] \in \overline{aco}\mathcal{D}_0(\mathcal{A})$ *if and only if given* $\epsilon > 0$, *there exist a set* $\{\alpha_k\}_{k=1}^n$ *of positive scalars and* $\{[L_k]\}_{k=1}^n \subset \mathcal{D}_0(\mathcal{A})$ *such that* $\sum_{k=1}^n \alpha_k = 1$, *and* $\|[L] - \sum_{k=1}^n \alpha_k [L_k]\| < \epsilon$.

LEMMA 3.2. *If a dual algebra* $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ *has property* $D_{0,\gamma-\theta}, 0 \le \theta < \gamma$, *then* $\mathcal{A}$ *has property* $D_{\theta,\gamma}$.

LEMMA 3.3. *Suppose* $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ *is a dual algebra that has property* $D_{\theta,\gamma}$ *for some* $0 \le \theta < \gamma$. *Then for every positive integer* n, *the dual algebra* $\mathcal{M}_n(\mathcal{A})$ *has property* $D_{\theta,\frac{\gamma}{n^2}}$.

Let n be a cardinal number satisfying $1 \le n \le \aleph_0$. We denote by $\tilde{\mathcal{H}}_n$ the Hilbert space consisting of the direct sum of n copies of $\mathcal{H}$ and by $T^{(n)}$ the n-fold ampliation of T acting on $\tilde{\mathcal{H}}_n$ defined by

$$T^{(n)}(x_1 \oplus \cdots \oplus x_n) = Tx_1 \oplus \cdots \oplus Tx_n. \tag{6}$$

Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra. We define $\mathcal{A}^{(n)} = \{\overbrace{A \oplus \cdots \oplus A}^{ncopies} : A \in \mathcal{A}\}$. Then $\mathcal{A}^{(n)}$ is indeed a dual algebra on $\tilde{\mathcal{H}}_n$. For each T in $\mathcal{L}(\mathcal{H})$, it is clear that $(\mathcal{A}_T)^{(n)} = \mathcal{A}_{T^{(n)}}$.

LEMMA 3.4 ([3], PROPOSITION 2.5). *If* $1 \le n \le \aleph_0$ *and* $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ *is a dual algebra, then* $\mathcal{A}^{(n)}$ *is a dual algebra which is isometrically isomorphic to* $\mathcal{A}$ *via the linear transformation* $\Phi : \mathcal{A} \to \mathcal{A}^{(n)}$. *Moreover* $\Phi = \phi^*$, *where* $\phi : Q_{\mathcal{A}^{(n)}} \to Q_{\mathcal{A}}$ *is also an isometric isomorphism (onto* $Q_{\mathcal{A}}$*), and if* $\tilde{x} = (x_0, x_1, \cdots)$ *and* $\tilde{y} = (y_0, y_1, \cdots)$ *are vectors in* $\mathcal{H}^{(n)}$, *then* $\phi([\tilde{x} \otimes \tilde{y}]_{Q_{\mathcal{A}^{(n)}}}) = \sum_{0 \le i < n} [x_i \otimes y_i]_{Q_{\mathcal{A}}}$.

THEOREM 3.5. *Suppose that $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and n is a positive integer such that $\mathcal{A}^{(n)}$ has property $D_{\theta,\gamma}$ for some $\gamma > \theta \geq 0$. Also suppose that $\mathcal{A}^{(n)}$ is in $A_{.0}$, then $\mathcal{A}$ has property $D_{\theta+\frac{n-1}{n},\gamma-\theta}$.*

Proof. We have seen that $\mathcal{A}^{(n)}$ is a dual algebra. It suffices to prove that $\mathcal{D}_{\theta+\frac{n-1}{n}}(\mathcal{A})$ contains the open ball in $Q_{\mathcal{A}}$ of radius $\gamma-\theta$ centered at the origin. Thus let $[L] \in Q_{\mathcal{A}}$ satisfy $\|[L]\| < \gamma-\theta$, and write $\widetilde{[L]} = \phi^{-1}([L])$, so $\|\widetilde{[L]}\| < \gamma - \theta < \gamma$. Since $\mathcal{A}^{(n)}$ has property $D_{\theta,\gamma}$, by definition, there exist sequences $\{\tilde{x}^{(i)}\}_{i=1}^{\infty}$ and $\{\tilde{y}^{(i)}\}_{i=1}^{\infty}$ in $\mathcal{H}^{(n)}$ which converge weakly to zero such that

$$\limsup_{i} \|[\tilde{L}] - [\tilde{x}^{(i)} \otimes \tilde{y}^{(i)}]\| \leq \theta \tag{7}$$

$$\|\tilde{x}^{(i)}\| \leq 1, \|\tilde{y}^{(i)}\| \leq 1, \quad i = 1, 2, \cdots . \tag{8}$$

For each $i \in \mathbb{N}$ we write $\tilde{x}^{(i)} = (x_1^{(i)}, \cdots, x_n^{(i)}), \tilde{y}^{(i)} = (y_1^{(i)}, \cdots, y_n^{(i)})$, and from (7) and lemma 3.4 we have

$$\limsup_{i} \|[L] - \sum_{j=1}^{n} [x_j^{(i)} \otimes y_j^{(i)}]\| \leq \theta. \tag{9}$$

Since $\|[x \otimes y]\| \leq \|x\| \cdot \|y\|$, the Schwarz inequality and (8) yield

$$\begin{aligned} \sum_{j=1}^{n} \|[x_j^{(i)} \otimes y_j^{(i)}]\| &\leq \sum_{j=1}^{n} \|x_j^{(i)}\| \cdot \|y_j^{(i)}\| \\ &\leq \|\tilde{x}^{(i)}\| \cdot \|\tilde{y}^{(i)}\| \leq 1, \end{aligned} \tag{10}$$

for each $i \in \mathbb{N}$.

Next we choose j_i to satisfy $1 \leq j_i \leq n$ and

$$\|[x_{j_i}^{(i)} \otimes y_{j_i}^{(i)}]\| = \max_{1 \leq j \leq n} \|[x_j^{(i)} \otimes y_j^{(i)}]\|, \qquad i = 1, 2, \cdots . \tag{11}$$

Considering the two cases $\|[x_{j_i}^{(i)} \otimes y_{j_i}^{(i)}]\| \leq \frac{1}{n}$ and $\|[x_{j_i}^{(i)} \otimes y_{j_i}^{(i)}]\| > \frac{1}{n}$ separately, we see easily from (9), (10) and (11) in both cases that

$$\limsup \|[L] - [x_{j_i}^{(i)} \otimes y_{j_i}^{(i)}]\| \leq \theta + \frac{n-1}{n}, \qquad i = 1, 2, \cdots .$$

Since for each $i \in \mathbb{N}, 1 \le j_i \le n$, it follows that there exists an integer j_0 such that $j_i = j_0$ for infinitely many values of i. Thus we may drop down to a subsequence $\{[x_{j_0}^{(i_k)} \otimes y_{j_0}^{(i_k)}]\}_{k=1}^{\infty}$ such that $\limsup_k \|[L] - [x_{j_0}^{(i_k)} \otimes y_{j_0}^{(i_k)}]\|$ exists and is less than or equal to $\theta + \frac{n-1}{n}$. Furthermore, it is immediate from (8) that $\|x_{j_0}^{(i_k)}\| \le 1$ and $\|y_{j_0}^{(i_k)}\| \le 1$ for all k. Finally, suppose $z \in \mathcal{H}$, and let $\tilde{z}$ in $\mathcal{H}^{(n)}$ be the vector with z as its only nonzero component, sitting in the j_0th slot. Then it follows easily from $\mathcal{A}^{(n)} \in A_{\cdot 0}$ and the fact that $\phi([\tilde{x}^{(i_k)} \otimes \tilde{z}]) = [x_{j_0}^{(i_k)} \otimes z]$ that $\|[x_{j_0}^{(i_k)} \otimes z]\| \to 0$. Hence it follows from $\|[x_{j_0}^{(i_k)} \otimes z]\| \to 0, \quad \|\langle 1, [L]\rangle\| \le \|[L]\|$ and $\mathcal{A}$ is a dual algebra that the sequences $\{x_{j_0}^{(i_k)}\}_{k=1}^{\infty}$ must converge weakly to zero in $\mathcal{H}$. Since the weak convergence to zero of $y_{j_0}^{(i_k)}$ follows similarly, we have shown that $[L] \in \mathcal{D}_{\theta + \frac{n-1}{n}}(\mathcal{A})$, so $\mathcal{A}$ has property $D_{\theta + \frac{n-1}{n}, \gamma - \theta}$.

PROPOSITION 3.6. *Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra. Then $\mathcal{A}^{(\aleph_0)}$ is a dual algebra with property $D_{0,1}$.*

Proof. Since $\mathcal{A}$ is a dual algebra, by [5, Proposition 3.9], $\mathcal{A}^{(\aleph_0)}$ is a dual algebra with $X_{0,1}$. Hence, clearly $\mathcal{A}^{(\aleph_0)}$ is a dual algebra with property $D_{0,1}$, by [5,Definition 2.7].

THEOREM 3.7. *Suppose that $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and n is a positive integer such that $\mathcal{A}^{(n)}$ has property $D_{0,\gamma-\theta}$ for some $0 \le \theta < \gamma$. Also suppose that $\mathcal{A}^{(n)}$ is in class $A_{\cdot 0}$, then $\mathcal{A}$ has property $D_{\theta + \frac{n-1}{n}, \gamma - \theta}$.*

Proof. Since $\mathcal{A}^{(n)}$ is a dual algebra, by lemma 3.2, $\mathcal{A}^{(n)}$ has property $D_{\theta,\gamma}$. Hence $\mathcal{A}$ has property $D_{\theta + \frac{n-1}{n}, \gamma - \theta}$, by theorem 3.5.

COROLLARY 3.8. *Under the hypotheses of theorem 3.7 with $\gamma - 2\theta > \frac{n-1}{n}$, the dual algebra $\mathcal{M}_n(\mathcal{A})$ has property $D_{\theta + \frac{n-1}{n}, \frac{(\gamma-\theta)}{n^2}}$.*

Proof. It is clear from lemma 3.3 and theorem 3.7.

REMARK 3.9. *Suppose that* $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ *is a dual algebra. Then* $\mathcal{A}$ *does not have property* $(\mathcal{A}_1)$ *though it has properties* $D_{\theta,\gamma}$ *for some* $0 \leq \theta < \gamma$.

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